The Parisian Traveller Problem

Thomas Bellitto[1] Johanne Cohen[2] Bruno Escoffier[1]
Minh-Hang Nguyen[3] Mikaël Rabie[3]

Tuesday, December 9, 2024

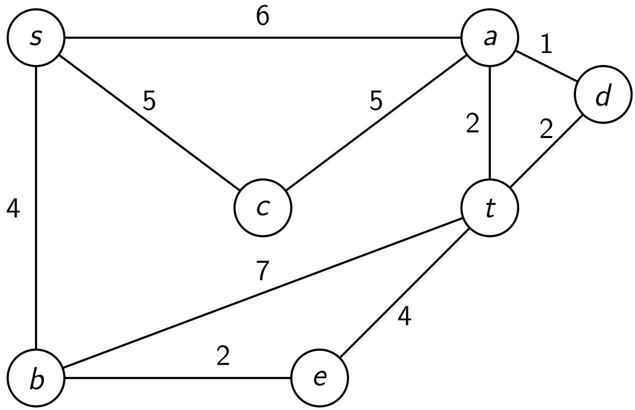
1 : Sorbonne Université, LIP6, Paris, France
2 : Université Paris-Saclay, LISN, France
3 : Université Paris Cité, IRIF, France

The Canadian Traveller Problem

- A traveller wants to go from a town s to a town t.
- It's in Canada and up to k of the roads shown on the map may be unusable.
- The traveller only learns if a road is unusable when they reach an incident vertex.

How much time does the traveller need in the worst case ? What should his strategy be ?

Example



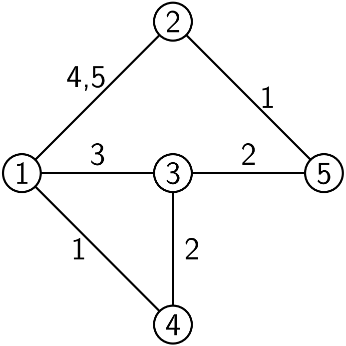
Temporal graph

Temporal graph

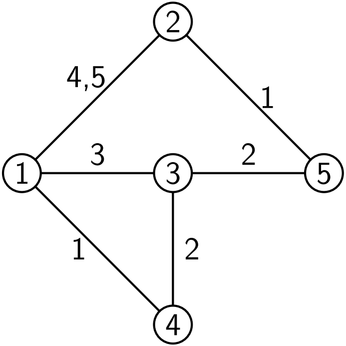
A *temporal graph* is defined by

- a set of vertices V
- a set of time edges. A time edge $(\{u, v\}, t)$ means that there is an edge between u and v at time t.
- a (strict) journey is a sequence $v_1, e_1, v_2, v_2, \ldots, e_{k-1}, v_k$ where e_i is an edge between v_i and v_{i+1} that appears at time $t_i > t_{i-1}$.

An example of temporal graph



An example of temporal graph



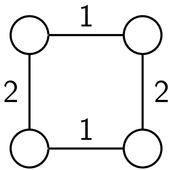
Not connected

$1 \not\rightarrow 5 \quad 2 \not\rightarrow 4$

$4 \not\rightarrow 5 \quad 5 \not\rightarrow 4$

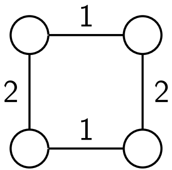
Example of problem : spanners

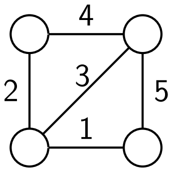
Some minimal connected graphs



Example of problem : spanners

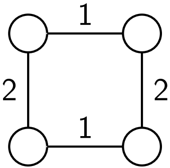
Some minimal connected graphs

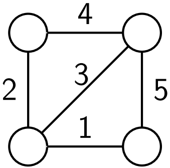


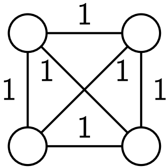


Example of problem : spanners

Some minimal connected graphs

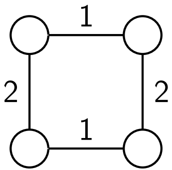


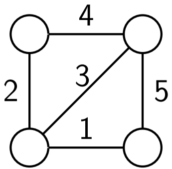


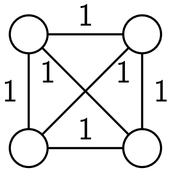


Example of problem : spanners

Some minimal connected graphs



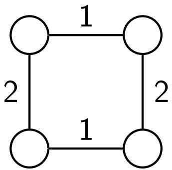


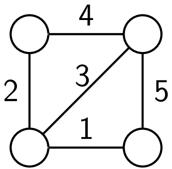


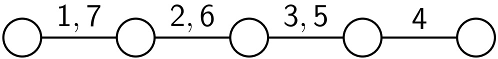
Not interesting

Example of problem : spanners

Some minimal connected graphs



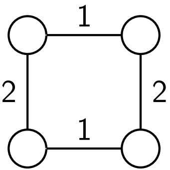


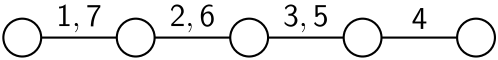


$2n - 3$ edges

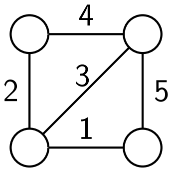
Example of problem : spanners

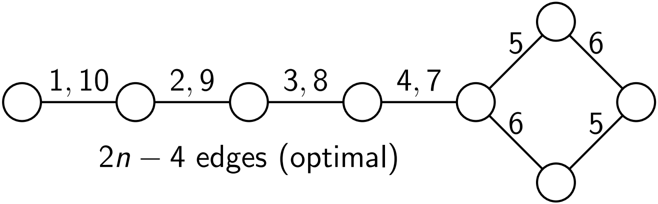
Some minimal connected graphs



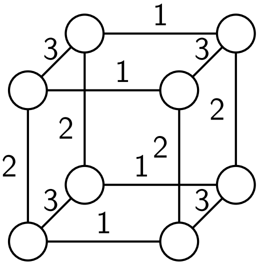


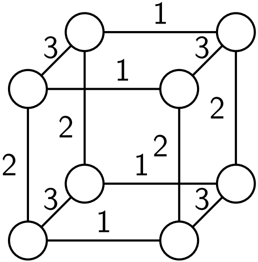
$2n - 3$ edges



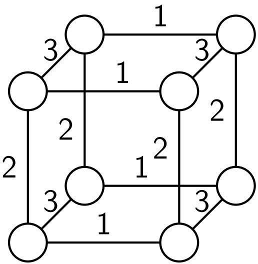


$2n - 4$ edges (optimal)


1
3
3
1
2
2
2
1
2
3
3
1



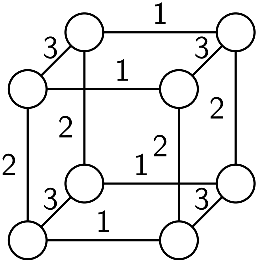
$\Theta(n \log(n))$ edges



$\Theta(n \log(n))$ edges

Theorem (Axiotis and Fotalis, 2016)

There are families of proper minimal connected temporal graphs with $\Theta(n^2)$ edges.



$\Theta(n \log(n))$ edges

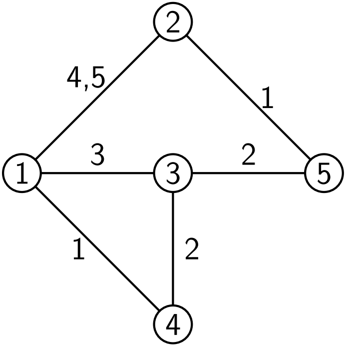
Theorem (Axiotis and Fotalis, 2016)

There are families of proper minimal connected temporal graphs with $\Theta(n^2)$ edges.

Conjecture (Casteigts)

Every complete proper temporal graphs contains a spanners of size $2n - 3$ or $2n - 4$.

Another example : connectivity augmentation

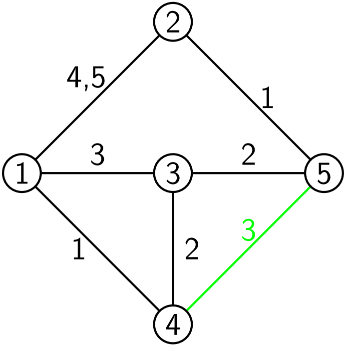


Not connected

$1 \not\to 5 \quad 2 \not\to 4$

$4 \not\to 5 \quad 5 \not\to 4$

Another example : connectivity augmentation

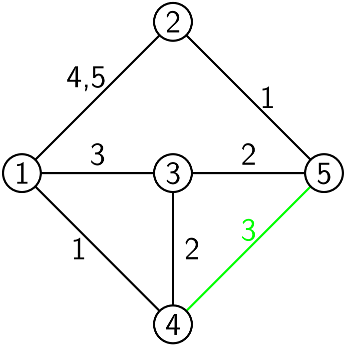


Not connected

$1 \not\to 5 \quad 2 \not\to 4$

$4 \not\to 5 \quad 5 \not\to 4$

Another example : connectivity augmentation



Not connected

$1 \not\to 5 \quad 2 \not\to 4$

$4 \not\to 5 \quad 5 \not\to 4$

Theorems (Bellitto, Bouton Popper, Escoffier, 2024+)

NP-hard even under many strong hypotheses.

The Uninformed Parisian Traveller Problem

Project

Study the Canadian Traveller Problem in temporal graphs.

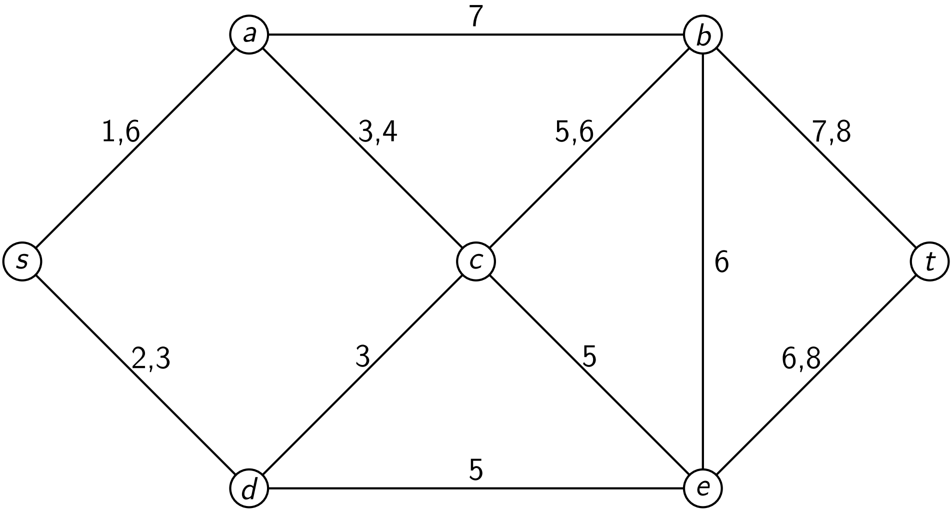
The Uninformed Parisian Traveller Problem

Project

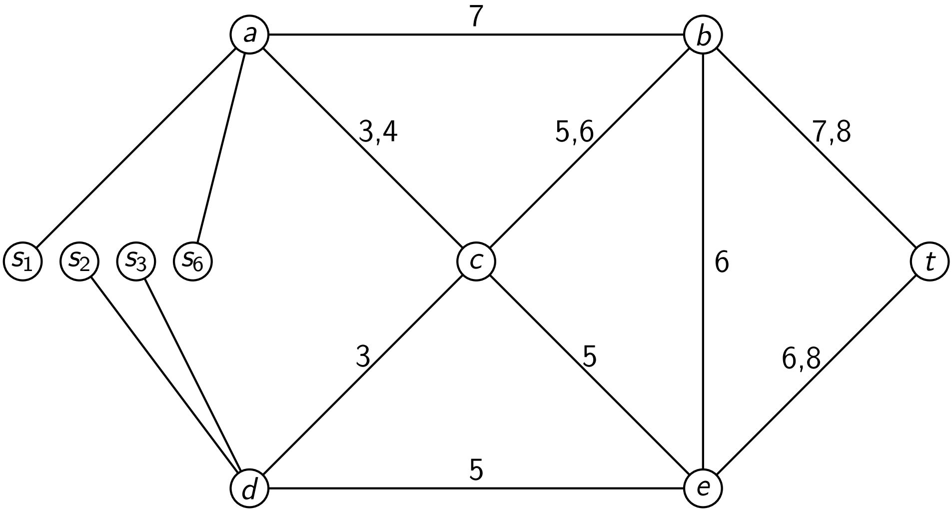
Study the Canadian Traveller Problem in temporal graphs.

- We want to go from s to t.
- Up to k time edges are missing.
- We only know if a time edge $(\{u, v\}, t)$ is missing if we are vertex u or v at time t.
- Earliest arrival ? Latest departure ? Shortest travel time ?

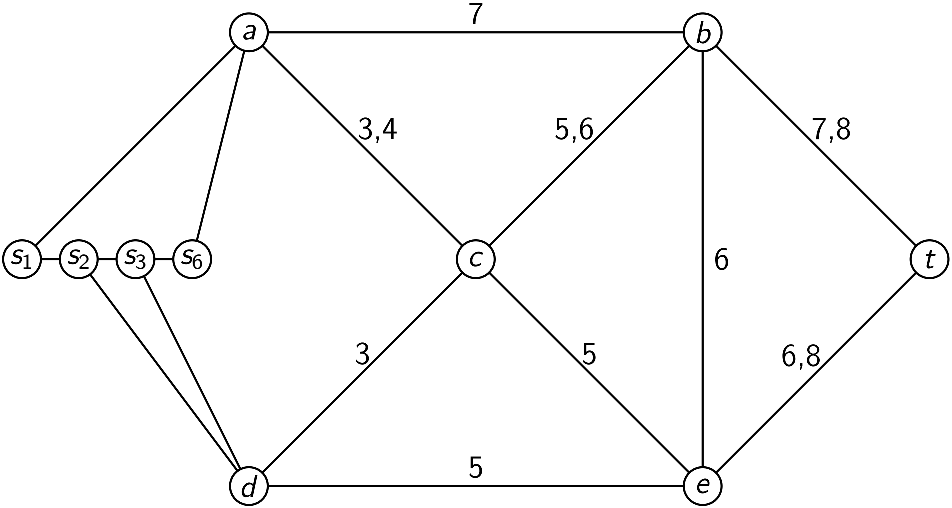
Example



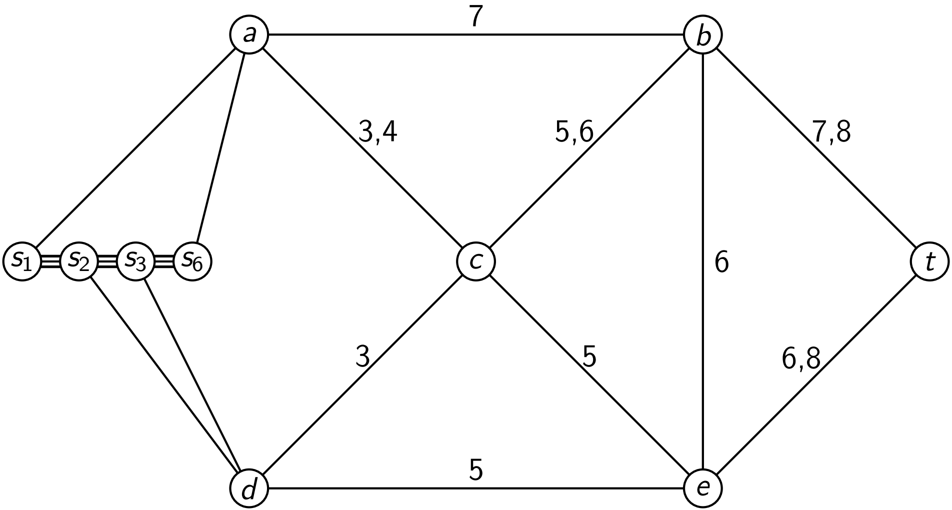
Example



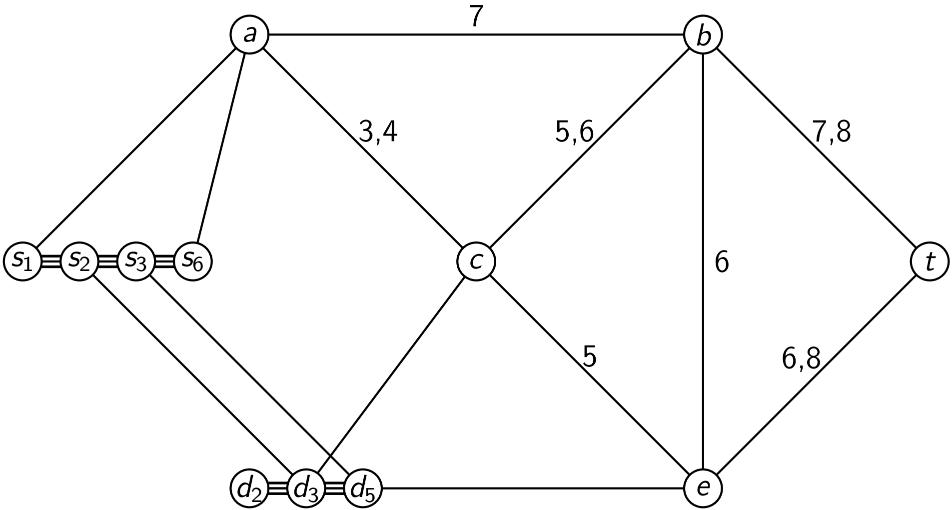
Example



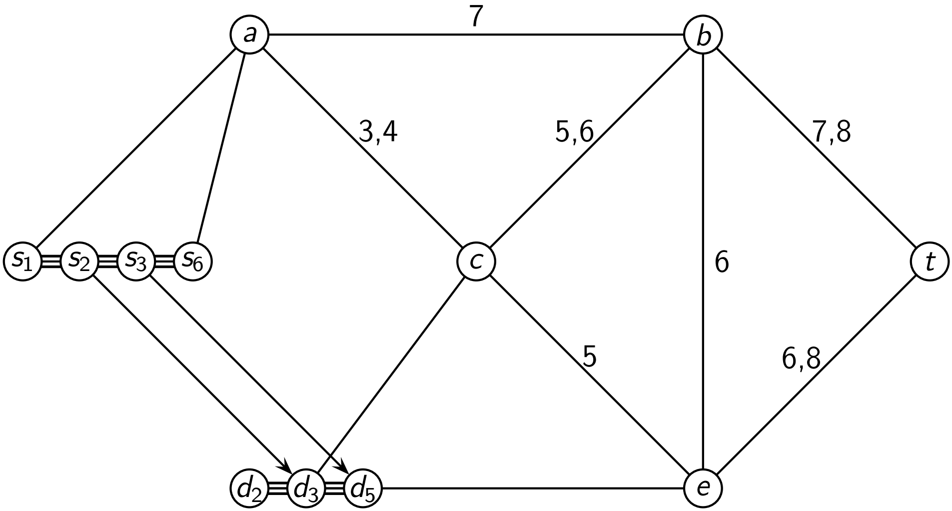
Example



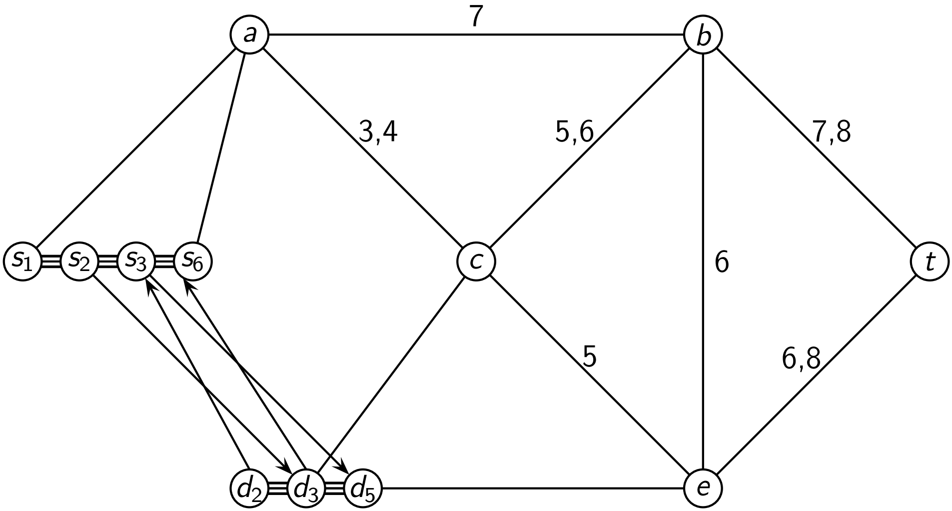
Example



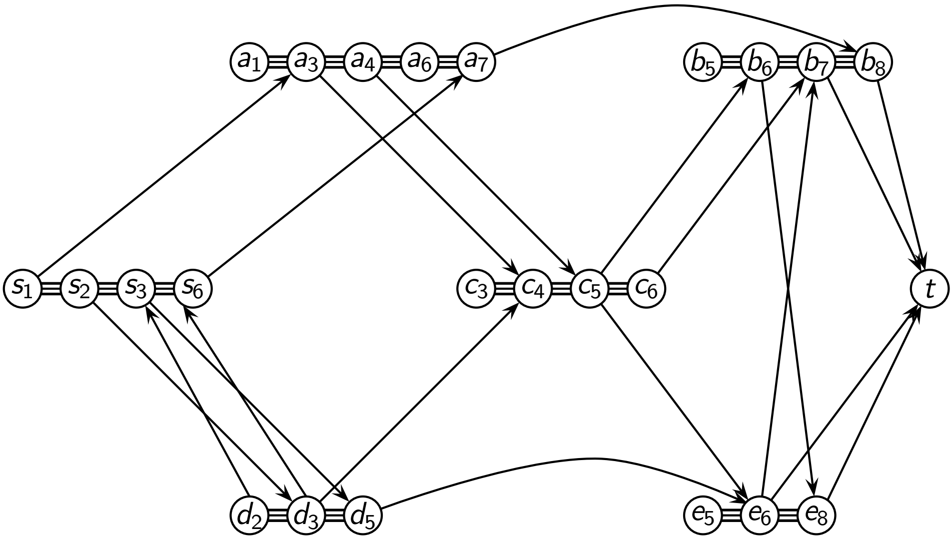
Example



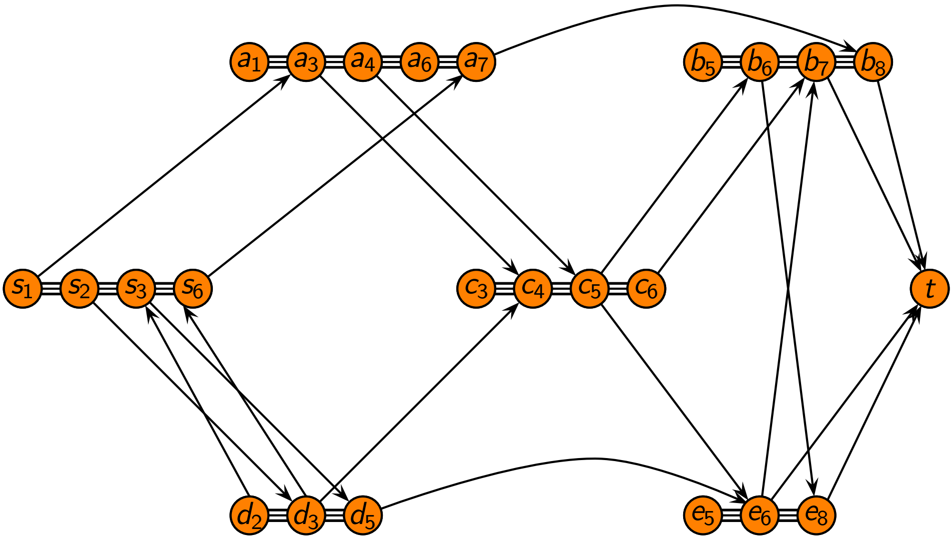
Example



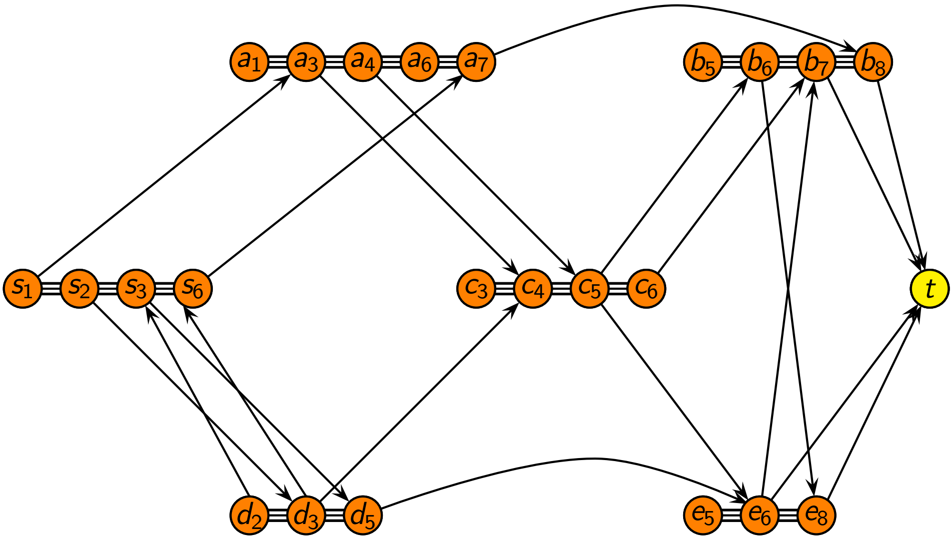
Example



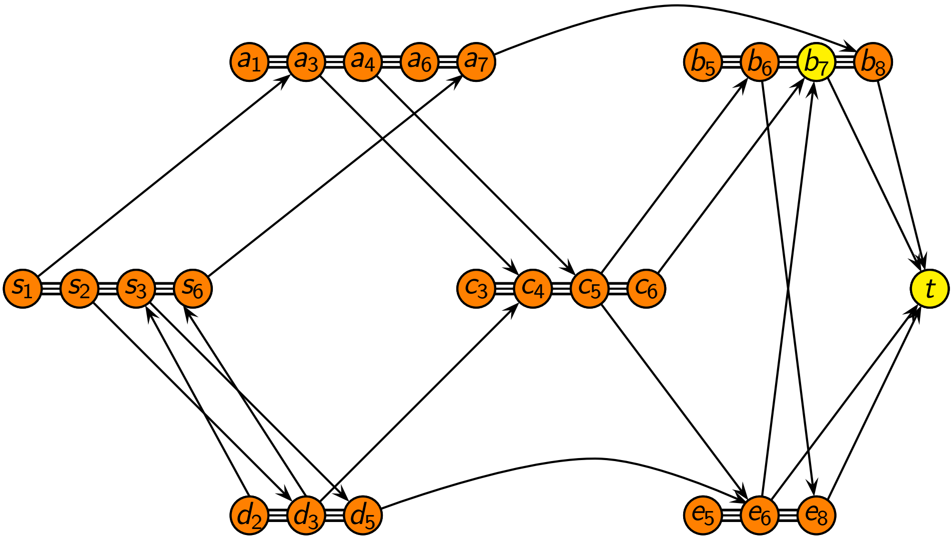
Example



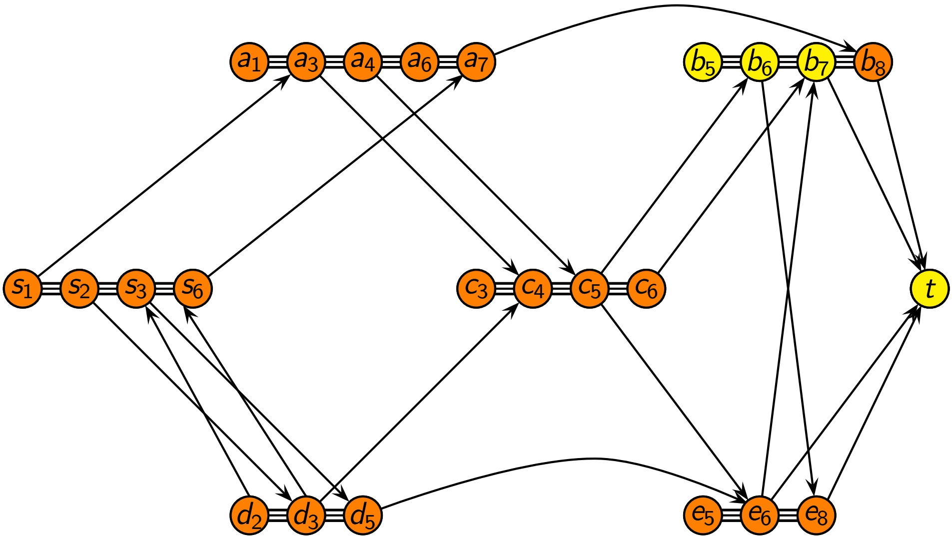
Example



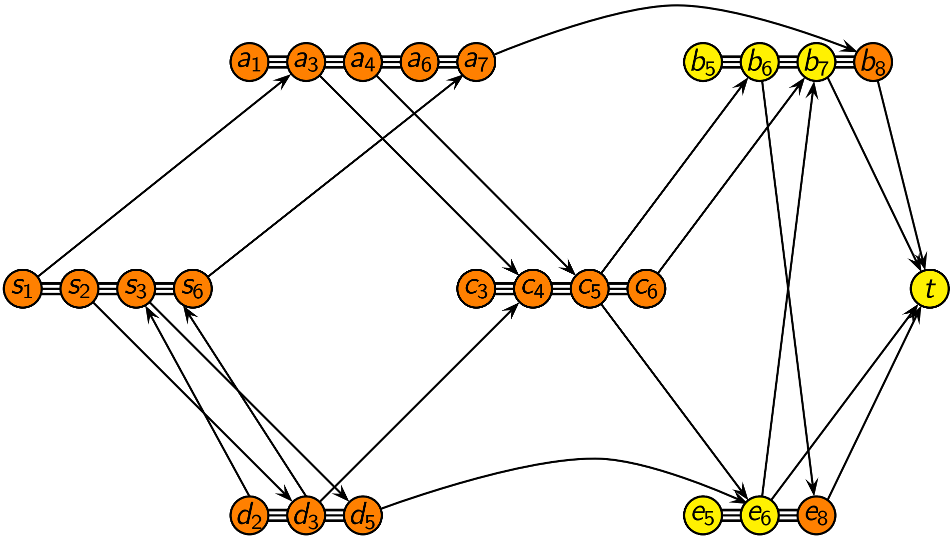
Example



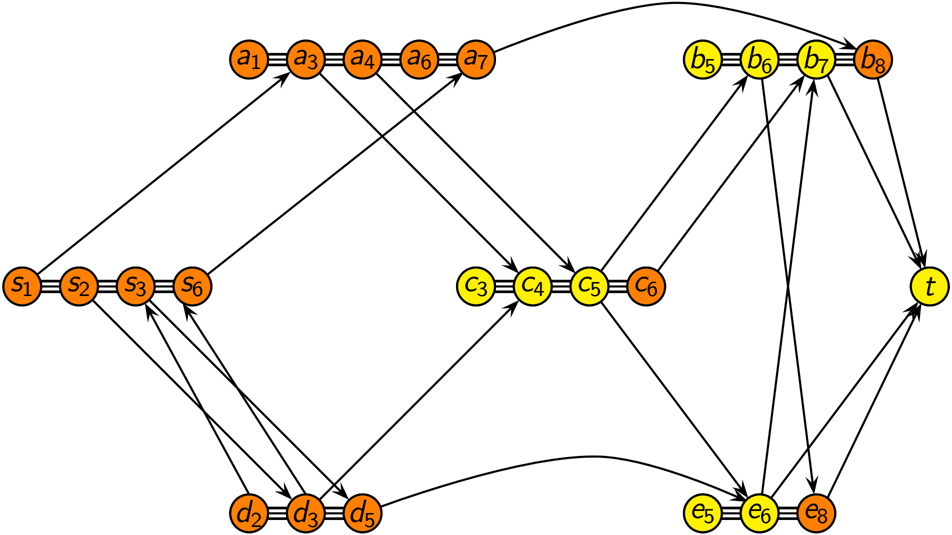
Example



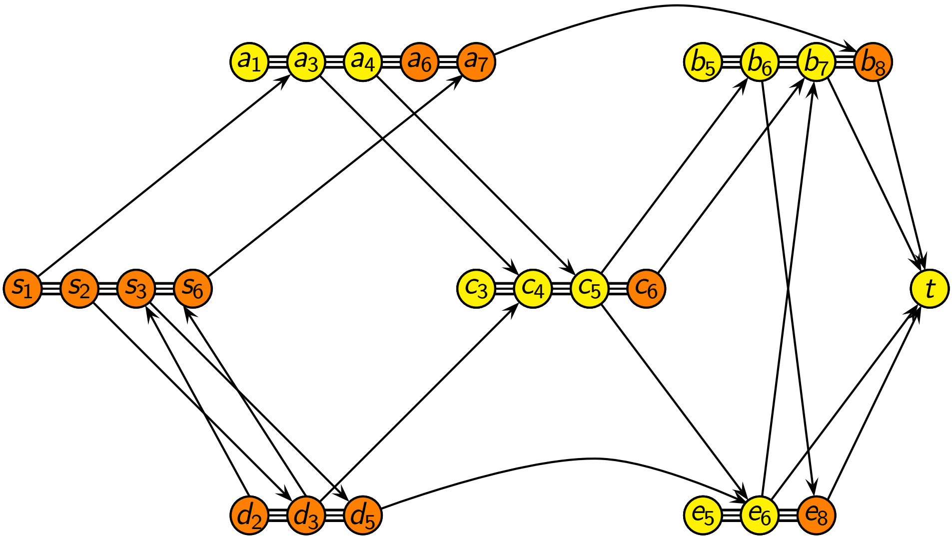
Example



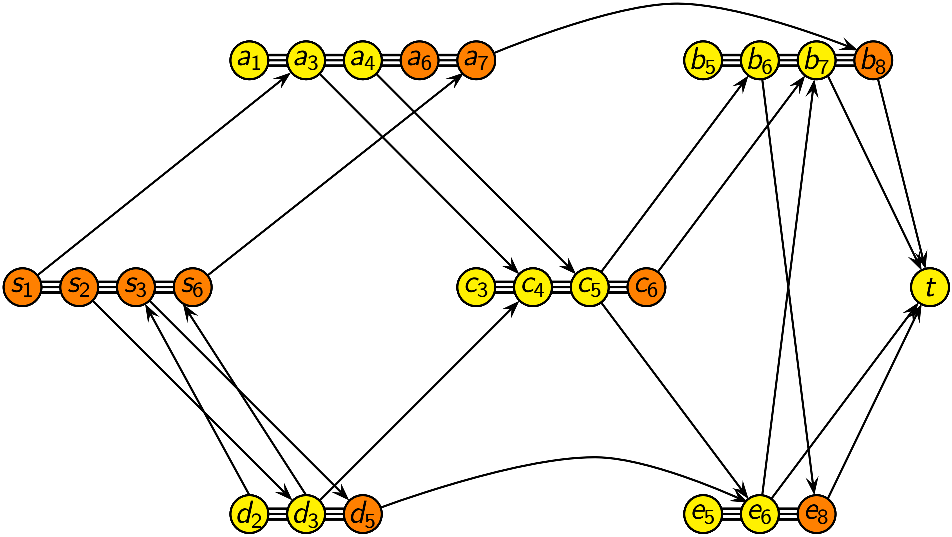
Example



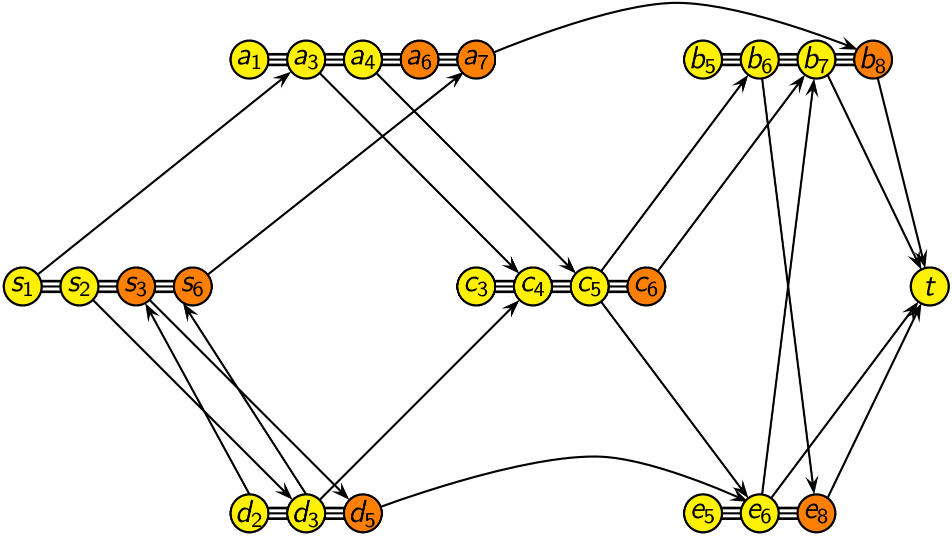
Example



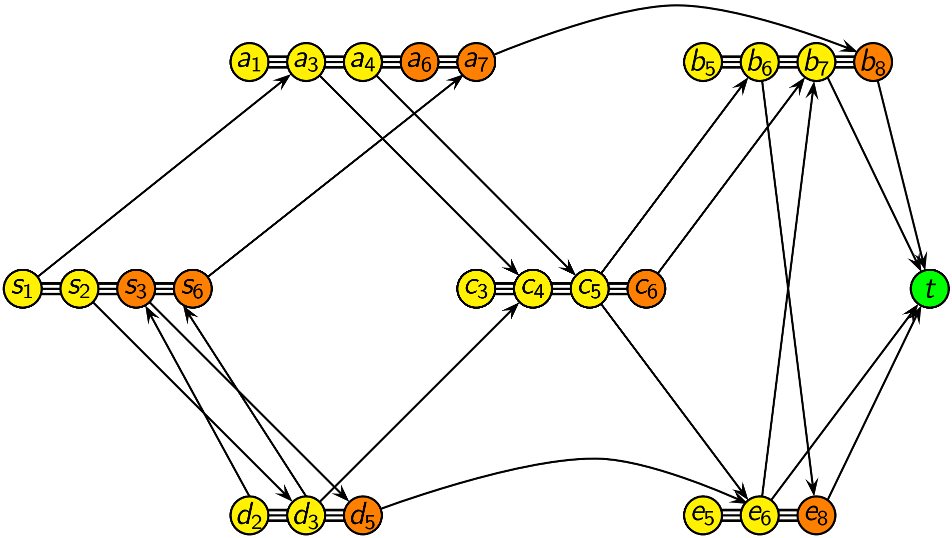
Example



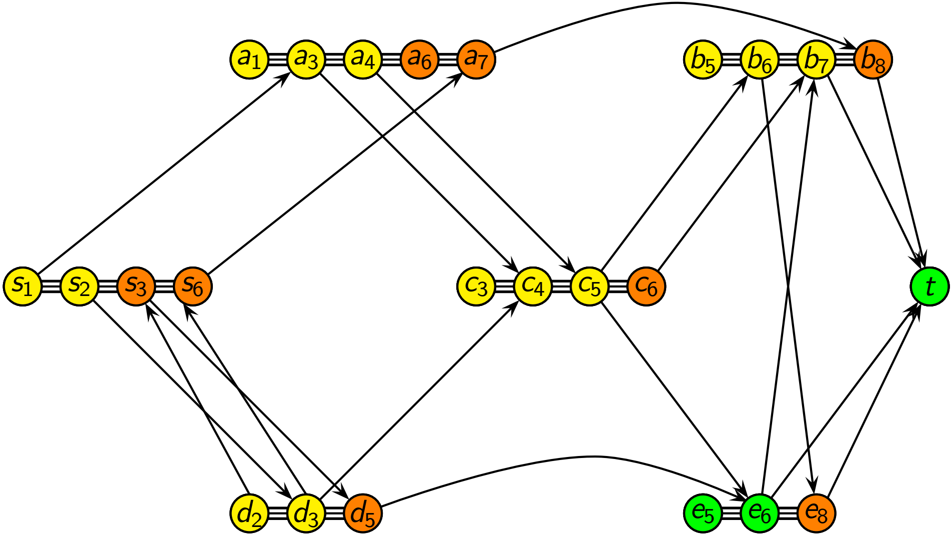
Example



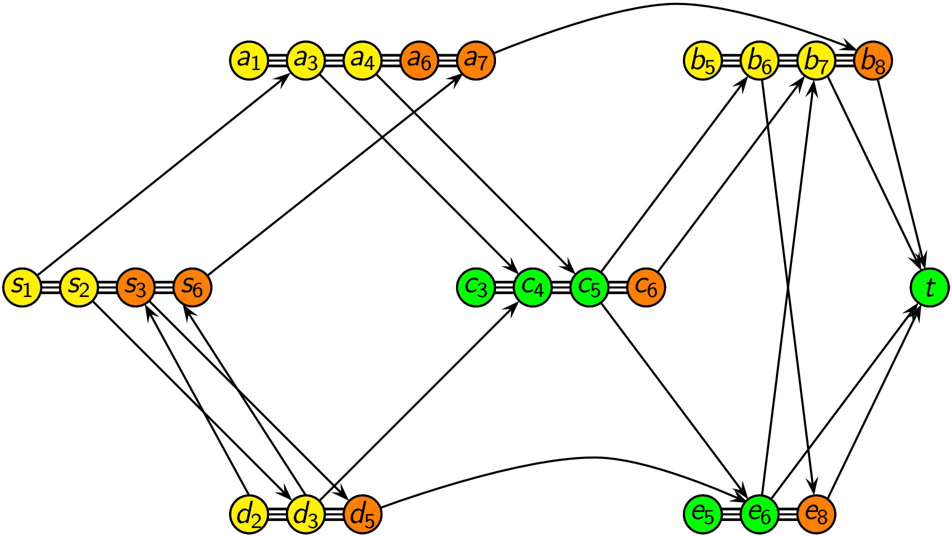
Example



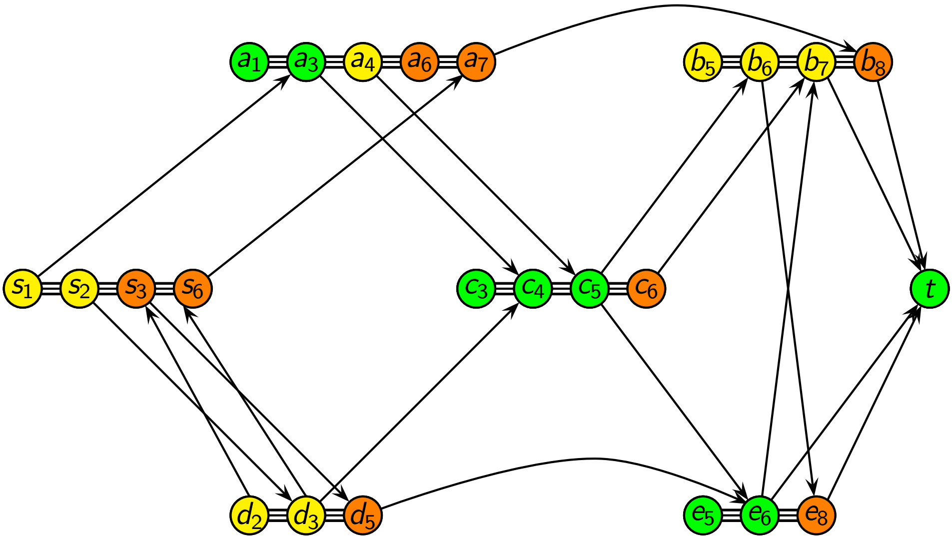
Example



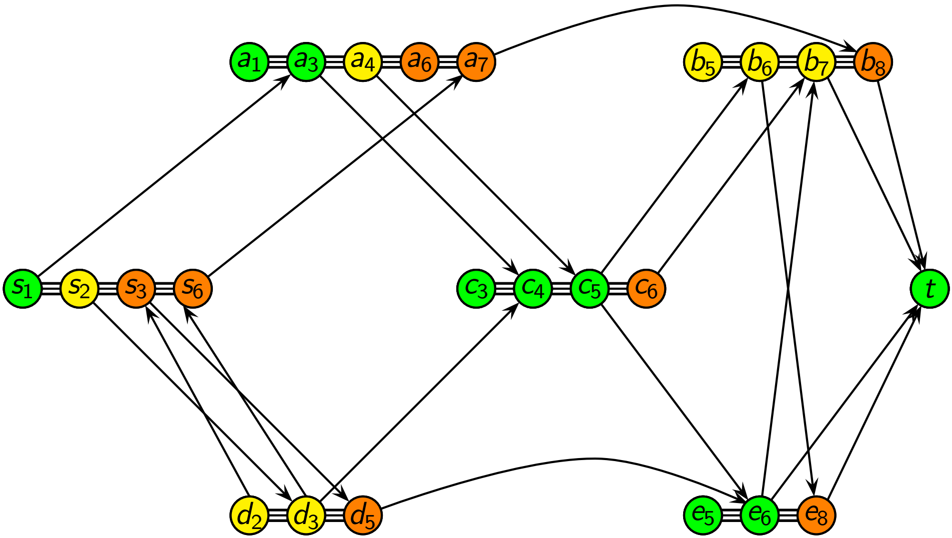
Example



Example



Example



Locally-informed Parisian Traveller Problem

What if

- Blocker can block all the edges between two vertices at once ?

Locally-informed Parisian Traveller Problem

What if

- Blocker can block all the edges between two vertices at once ? $\rightarrow$ Generalization of the static case

Locally-informed Parisian Traveller Problem

What if

- Blocker can block all the edges between two vertices at once ? $\rightarrow$ Generalization of the static case
- You know in advance which edges are blocked ?

Locally-informed Parisian Traveller Problem

What if

- Blocker can block all the edges between two vertices at once ? $\rightarrow$ Generalization of the static case
- You know in advance which edges are blocked ?
  $\rightarrow$ Shortest path

Locally-informed Parisian Traveller Problem

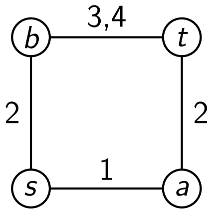
What if

- Blocker can block all the edges between two vertices at once ? $\rightarrow$ Generalization of the static case
- You know in advance which edges are blocked ? $\rightarrow$ Shortest path
- You know in advance about the edges incident to the current ?

Locally-informed Parisian Traveller Problem

What if

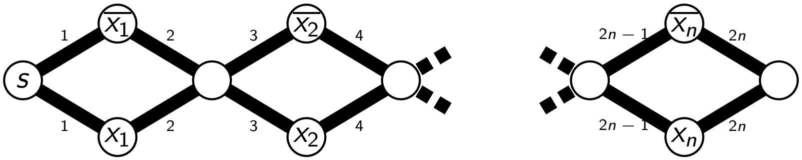
- Blocker can block all the edges between two vertices at once ? → Generalization of the static case
- You know in advance which edges are blocked ?
  → Shortest path
- You know in advance about the edges incident to the current ?
  Locally-informed Parisian Traveller Problem



Complexity

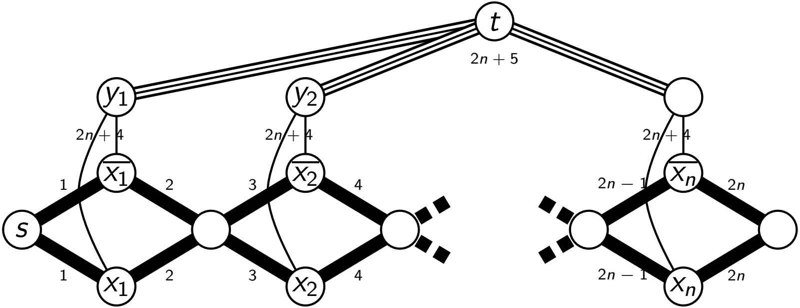
$\mathscr{F}$ formula with n variables and m clauses.

Breaker may block up to $2m$ edges.



bold edges = unbreakable

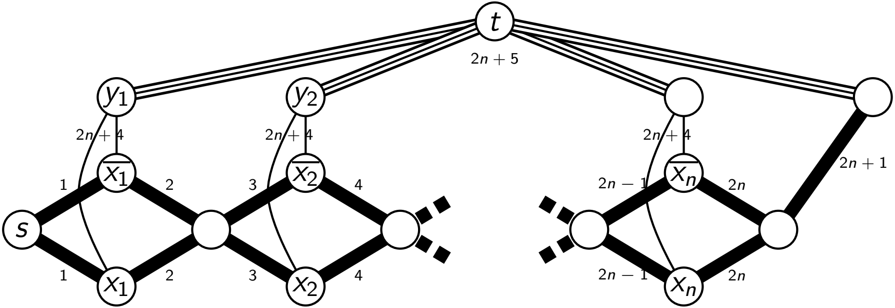
Complexity



bold edges = unbreakable

triple edges = breaker's max capacity

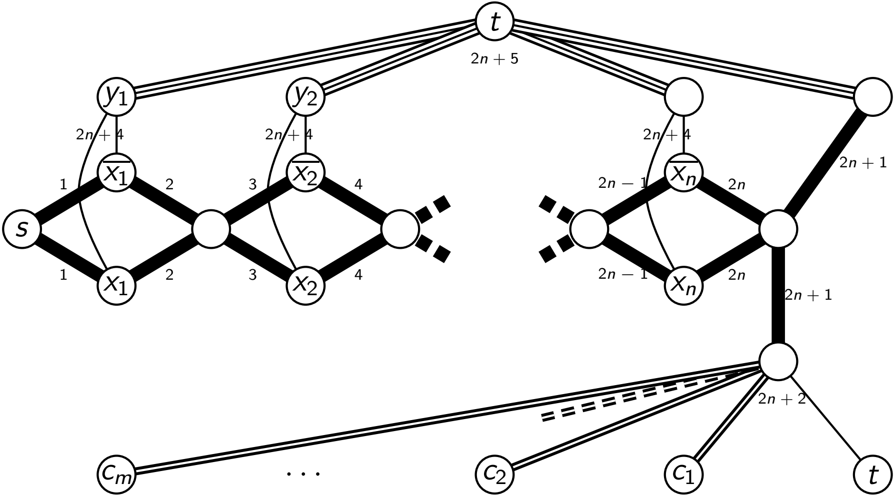
Complexity



bold edges = unbreakable

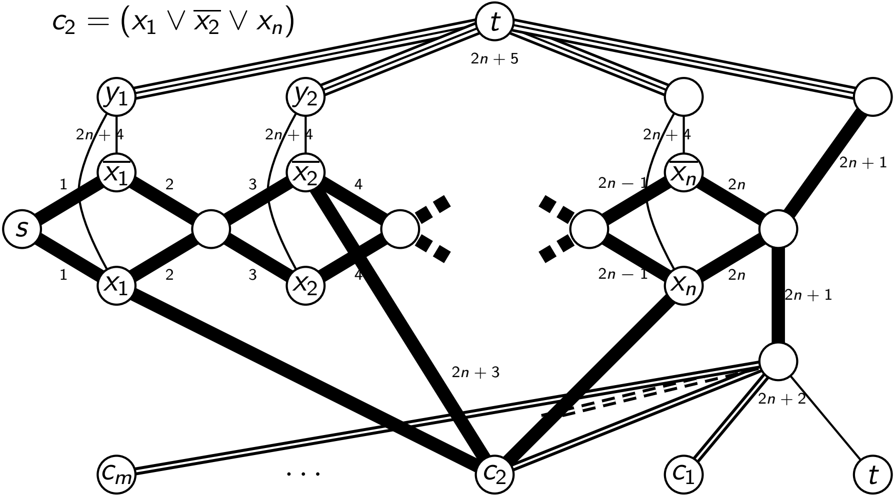
triple edges = breaker's max capacity

Complexity


t
2n + 5
y1
y2
2n + 4
2n + 4
2n + 4
x1
x2
xn
1
2
3
4
2n − 1
2n
2n + 1
s
1
2
3
4
2n − 1
2n
x1
x2
xn
2n + 1
2n + 2
cm
. . .
c2
c1
t

Complexity

$c_2 = (x_1 \vee \overline{x_2} \vee x_n)$



Fixed number of edges

Theorem

The Temporal Canadian Traveller Problem is

- polynomial if there can only be one blocked edge ;
- NP-hard as soon as there can be $k \geqslant 2$ blocked edges.

Fixed number of edges

Theorem

The Temporal Canadian Traveller Problem is

- polynomial if there can only be one blocked edge ;
- NP-hard as soon as there can be $k \geqslant 2$ blocked edges.

Theorem

The Static Canadian Traveller Problem is

- polynomial if there can only be one blocked edge ;
- NP-hard with $k \geqslant 4$ blocked edges.

PSPACE-completeness

- Every NP problem can be solved in polynomial space.

3-SAT :

Is $\mathscr{F} =$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

satisfiable ? Yes

PSPACE-completeness

- Every NP problem can be solved in polynomial space.

3-SAT :

Is $\mathscr{F} = \exists x_1, \exists x_2, \exists x_3$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

true ? Yes

PSPACE-completeness

- Every NP problem can be solved in polynomial space.

3-SAT :

Is $\mathscr{F} = \exists x_1, \forall x_2, \exists x_3$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

true ? Yes

PSPACE-completeness

- Every NP problem can be solved in polynomial space.

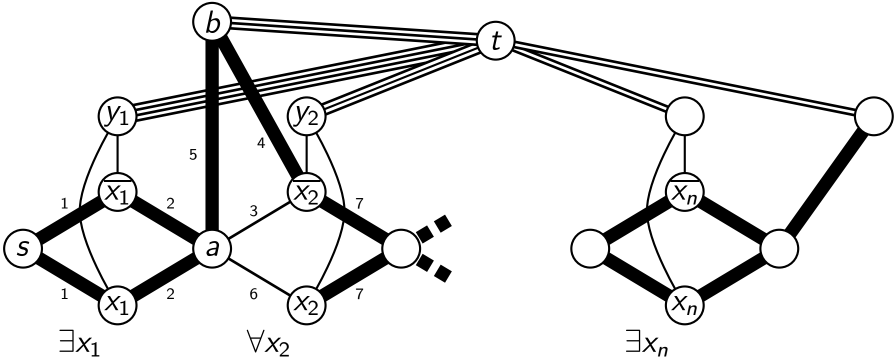
3-SAT :

Is $\mathscr{F} = \exists x_1, \exists x_2, \forall x_3$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$$

true ? No

Sketch of the proof

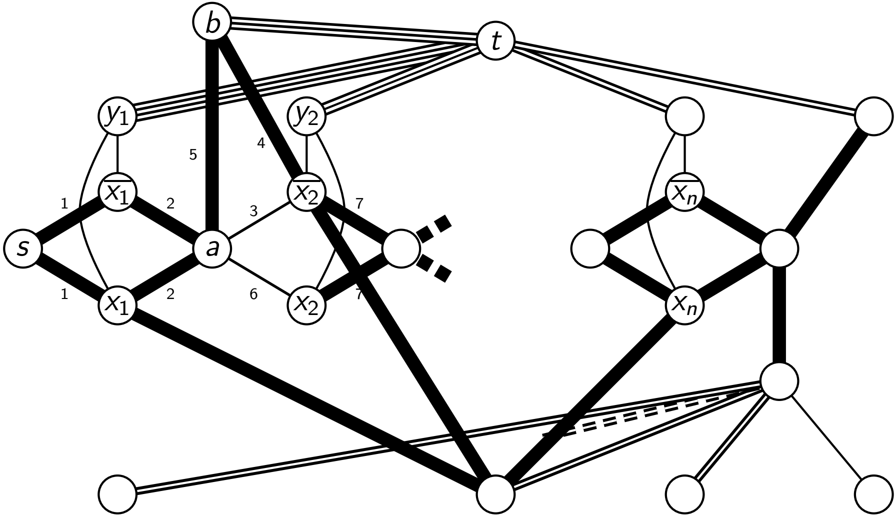


bold edges = unbreakable

Breaker has an aditionnal edge for every ∀ in the formula

multiple edges = breaker's remaining budget

Sketch of the proof



Result

Theorem

The Temporal Canadian Traveller Problem is PSPACE-complete.

Result

Theorem

The Temporal Canadian Traveller Problem is PSPACE-complete.

The proof does not hold with a bounded number of edges.

Result

Theorem

The Temporal Canadian Traveller Problem is PSPACE-complete.

The proof does not hold with a bounded number of edges. But the result most likely does not hold either since the problem is NP !

Result

Theorem

The Temporal Canadian Traveller Problem is PSPACE-complete.

The proof does not hold with a bounded number of edges. But the result most likely does not hold either since the problem is NP !

Thank you !