Connecting edge-colouring

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University of Southern Denmark

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Edge-colouring

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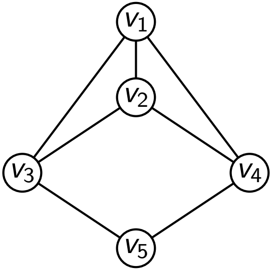
- k-edge-colouring: function $c : E \mapsto [1, k]$
- proper edge-colouring: $c(uv) \neq c(vw)$.



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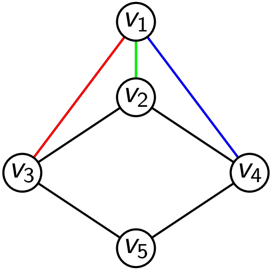

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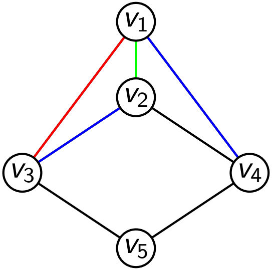

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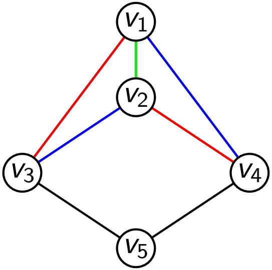

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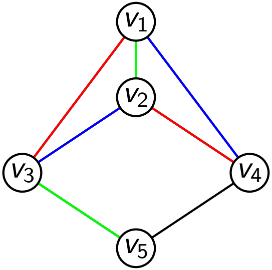

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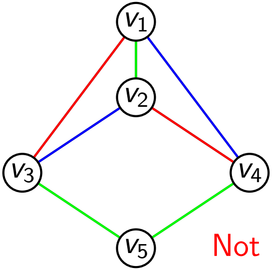

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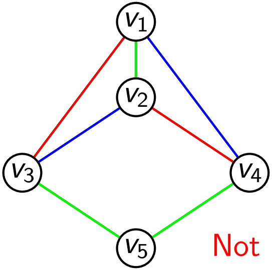

Not proper!

- Properly-coloured walk: does not use consecutively two edges of the same colour.

Context

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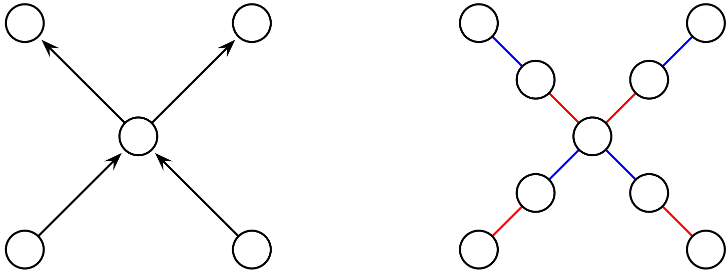

Proper connectivity

An edge-coloured graph $G_c = (V, E, c)$ is properly-connected if and only if there is a properly-coloured walk from every vertex to every other.

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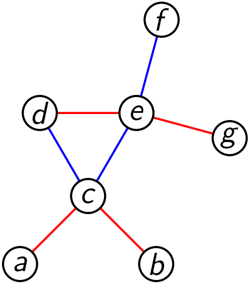

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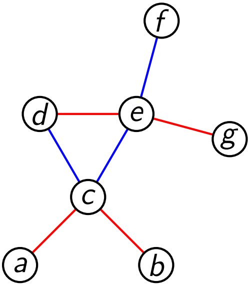

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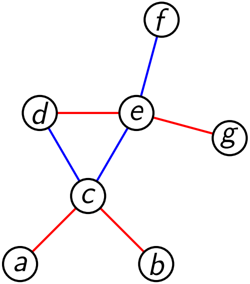

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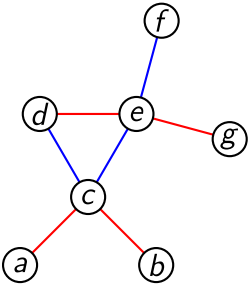

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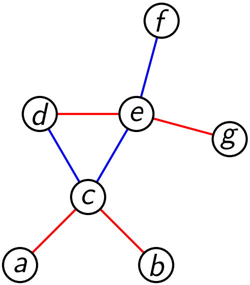

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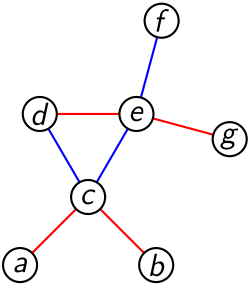

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Let $G = (V, E)$ be a connected graph. A k-edge-colouring is connecting iff $G_c = (V, E, c)$ is properly-connected.

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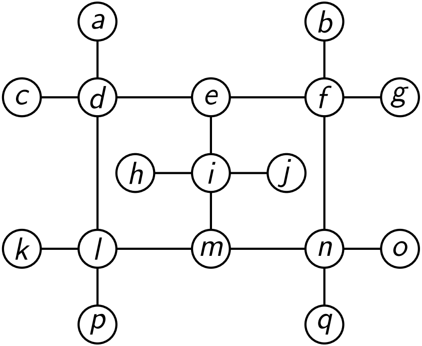

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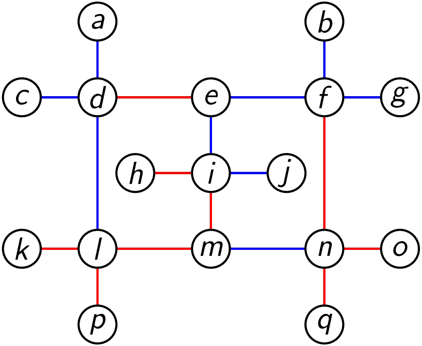
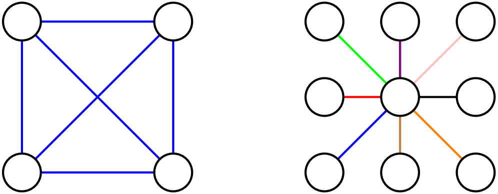


Trivial bounds

- We need between 1 and $n - 1$ colours.



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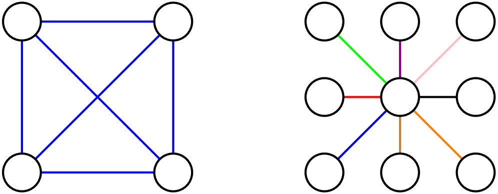

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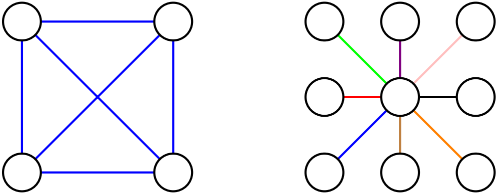

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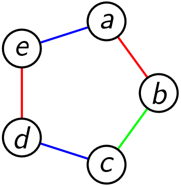

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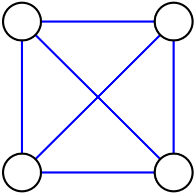

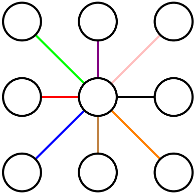

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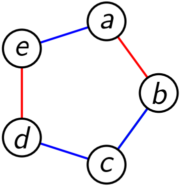

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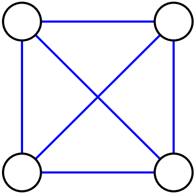
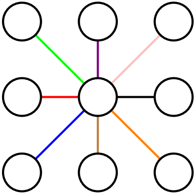

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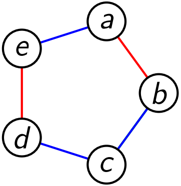

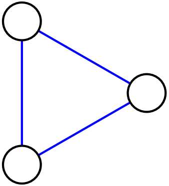

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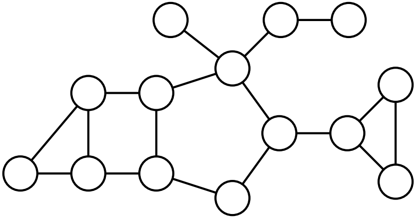

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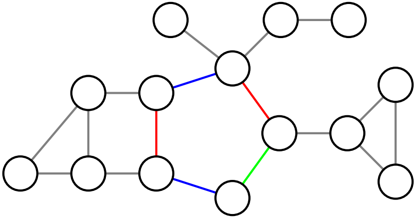

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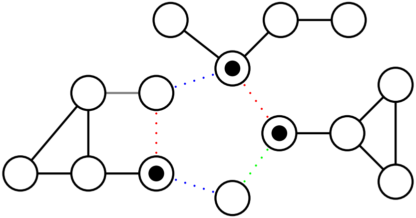

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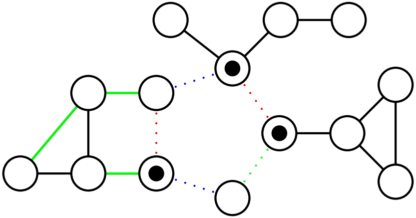

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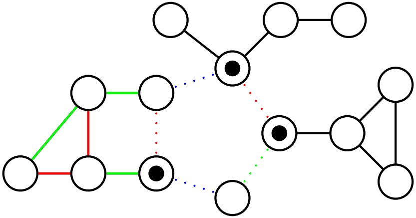

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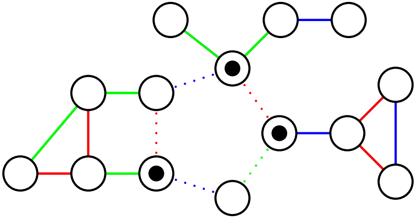

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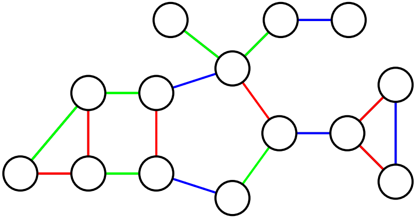

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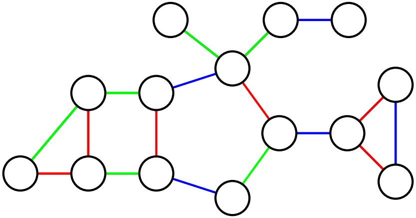
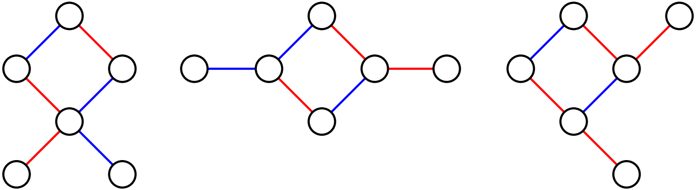


The graph we can connect with $k \geqslant 3$ colours are all the graphs except the trees who have a vertex of degree $> k$.

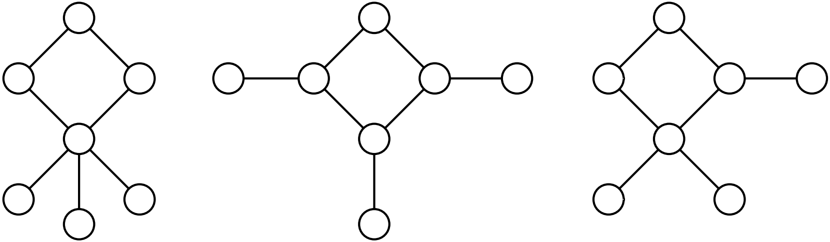
Bridges in bipartite graphs

All the paths between two vertices in a bipartite graph have same parity!



Bridges at even distance must have the different colours.
Bridges at odd distance must have the same colour.

Connectability of bipartite graphs



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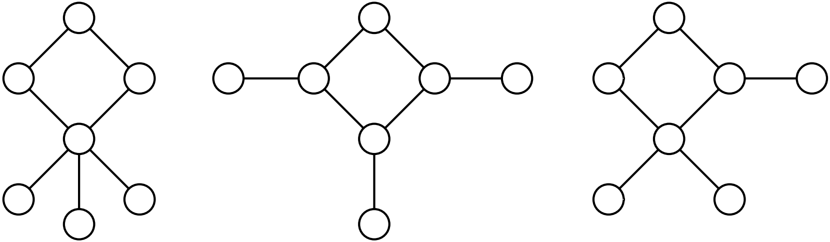

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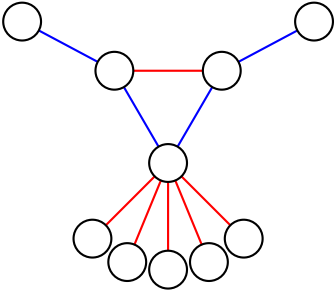
Theorem

A bipartite graph G can be connected with two colours if and only if it can be made 2-edge-connected by adding at most one edge.

Can we generalize it?



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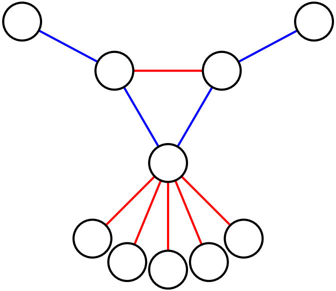

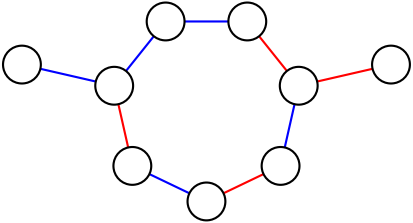

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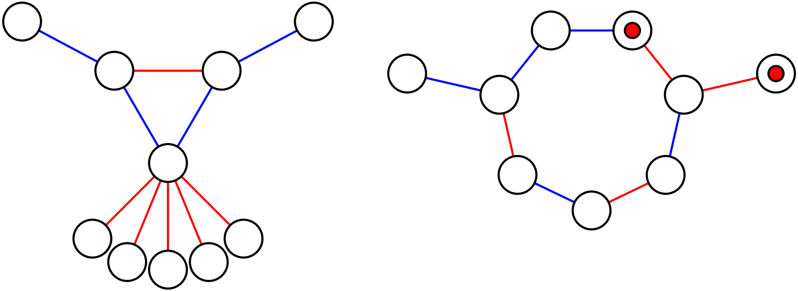
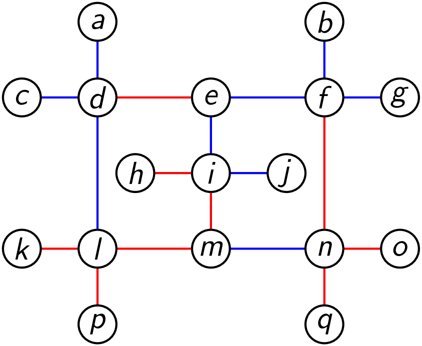


The power of odd cycles



Stubborn edges

The set $\mathscr{S}$ of stubborn edges is the set of the edges that belong to every odd cycle of the graph.

The $\mathscr{S}$-free components of the graph G are the connected components of $G \setminus \mathscr{S}$.

Theorem

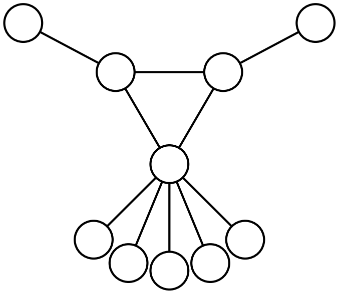
Let G be a non-bipartite 2-edge-coloured graph. Then, there exists an $\mathscr{S}$-free component $\mathcal{K}$ of G such that there is no properly-coloured walk between two vertices $u, v \notin \mathcal{K}$ that goes through a vertex $w \in \mathcal{K}$.

Existence of connecting 2-edge-colourings

Theorem

A connected non-bipartite graph G can be connected with two colours if and only if there exists a $\mathscr{S}$-free component $\mathcal{K}$ of G such that $G \setminus \mathcal{K}$ is empty or can be made 2-edge-connected by adding at most one edge.

Example of connectable graph



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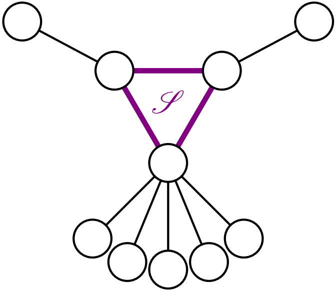

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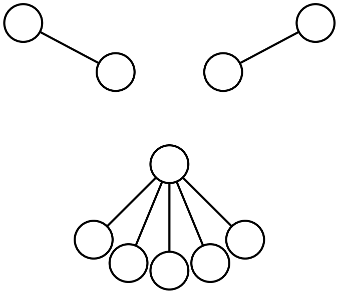

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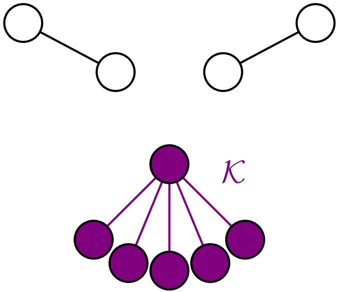

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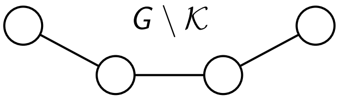

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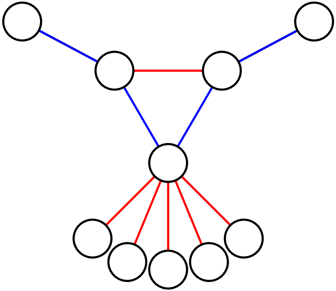

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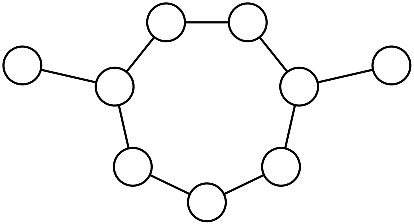

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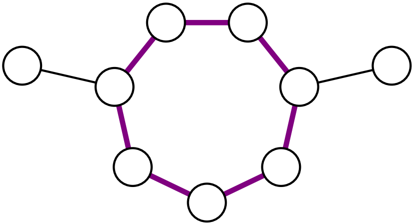

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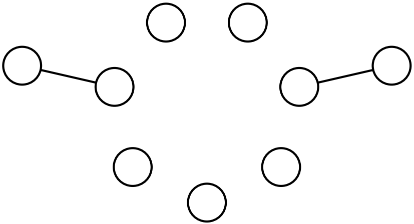

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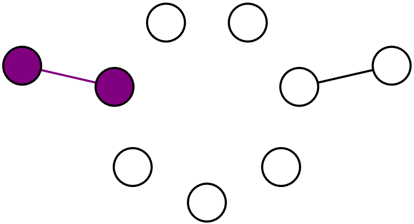

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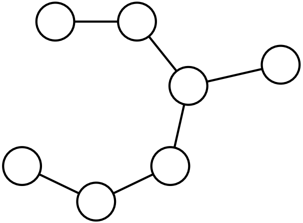

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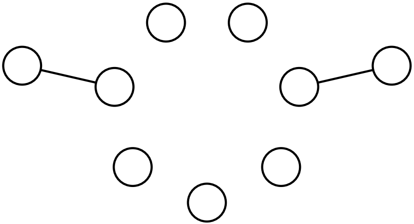
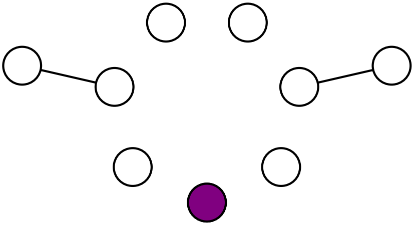
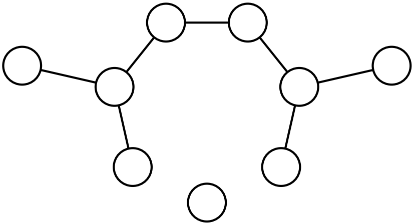


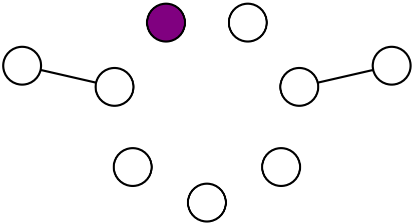
Example of non-connectable graph



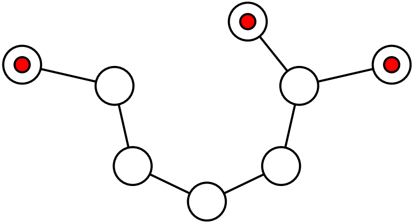
Example of non-connectable graph



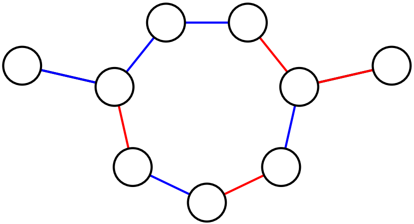
Example of non-connectable graph



Example of non-connectable graph



Example of non-connectable graph



Putting everything together

The minimum number of colours required for a connecting edge-colouring of a graph G is:

- 1 if G is complete;
- its maximum degree $\Delta(G)$ if G is a tree;
- 2 if G is bipartite and can be made 2-edge-connected by adding at most one edge;
- 2 if G is non-bipartite and contains an $\mathscr{S}$-free component $\mathcal{K}$ such that $G \setminus \mathcal{K}$ is empty or can be made 2-edge-connected by adding at most one edge;
- 3 otherwise

In every case, we can find an optimal connecting colouring in polynomial time.

Ideas for future work

- Stronger definitions of connectivity (cycles? spanning closed walk?)
- Connecting with trails or paths instead of walks?
- Stretch of the edge-colouring?
- Extension of partial edge-colouring?
- Study of directed graphs?

Thank you!