Graph Colouring Under Input Restrictions

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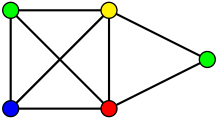
Graph Colouring



A colouring is an assignment of colours to the vertices of a graph such that no two adjacent vertices get the same colour.

A colouring using at most k colours is a k-colouring.

COLOURING
Instance: a graph G and an integer k.
Question: does G have a k-colouring?

k-COLOURING
Instance: a graph G.
Question: does G have a k-colouring?

Theorem (Lovász, 1973)

3-COLOURING *is* NP*-complete.*

How to deal with NP-completeness?

This can be done in several ways. For example:

- heuristics
- approximation algorithms
- exact algorithms
- parameterized algorithms
- algorithms for restricted inputs

Our main focus is on the last approach. That is, we exploit the structure of the input with an aim to develop efficient algorithms.

Justification:

1. In practice, problem inputs may exhibit structure
2. By focussing on the input structure we understand better what structure makes a problem computationally hard.

Examples

A planar graph is a graph that can be drawn in the plane without edge crossings (except in the end-vertices of edges).

Theorem (4-Colour Theorem, Appel & Haken, 1977)

Every planar graph is 4-colourable.

Hence 4-COLOURING is trivial for planar graphs.

Theorem (Dailey, 1980)

3-COLOURING *is* NP*-complete for planar graphs.*

Theorem (Grötzsch, 1959)

Every triangle-free planar graph is 3-colourable.

Hence, 3-COLOURING is trivial for triangle-free planar graphs.

Complexity Dichotomies

Ideally, we would like to determine for any given graph class $\mathcal{G}$ if COLOURING is polynomial-time solvable or NP-complete for $\mathcal{G}$.

This is too ambitious. So we can consider a family of graph classes that have some common property instead of all graph classes.

A dichotomy theorem states that for each $\mathcal{G}$ from such a family, a certain NP-complete problem is

- either polynomial-time solvable on inputs from $\mathcal{G}$, or
- stays NP-complete even for $\mathcal{G}$.

This requires a systematic approach.

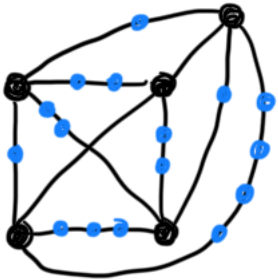
A Systematic Approach

Many natural and well-known graph classes can be characterized by some set of obstacles.

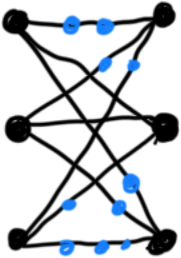
Theorem (Kuratowski's Theorem, 1930)

A graph is planar if and only if it does not contain a subdivided K_5 or subdivided $K_{3,3}$ as a subgraph.

a subdivided K_5



a subdivided $K_{3,3}$



Graph Classes

What are natural properties for a graph class to have?

Definition

A graph class is hereditary if it is closed under vertex deletion.

Definition

A graph class is monotone if it is closed under vertex deletion and edge deletion.

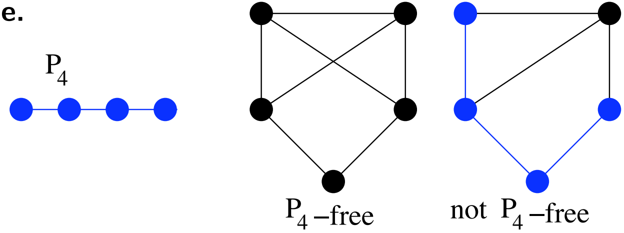
Example. The class of planar graphs is not only hereditary but even monotone.

Forbidding a Single Graph H

Definition

A graph H is a induced subgraph of a graph G if G can be modified into H by a sequence of vertex deletions; else G is H-free.

Example.



Definition

A graph H is a subgraph of a graph G if G can be modified into H by a sequence of vertex deletions and edge deletions; else G is H-subgraph-free.

An H-subgraph-free graph is also H-free, but reverse may be false.

Forbidding a Subgraph vs Induced Subgraph

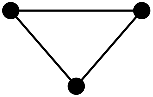
Example. The graph $3P_1$:

The class of $3P_1$-free graphs consists of all graphs with the property that there exists at least one edge inside each vertex triple.

Observation

A graph is $3P_1$-free if and only if its complement is triangle-free.

The triangle:



$3P_1$-free graphs form a rich graph class, for which many problems, such as determining the size of a largest clique, are NP-hard.

What about the class of $3P_1$-subgraph-free graphs?

This class consists of only four graphs: $(\emptyset, \emptyset)$, P_1, $2P_1$ and P_2.

Back To Our Systematic Approach

We can consider heredity graph classes or monotone graph classes.

However, even this is often too ambitious.

We can restrict to hereditary graph classes or monotone graph classes that are finitely defined, that is, the set of obstacles is finite.

Let $\{H_1, \ldots, H_p\}$ be a set of graphs, and let G be a graph.

Definition

G is $(H_1, \ldots, H_p)$-free if G is H_i-free for $i = 1, \ldots, p$.

Definition

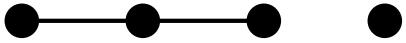
G is $(H_1, \ldots, H_p)$-subgraph-free if G is H_i-subgraph-free for $i = 1, \ldots, p$.

As a starting point we can first focus on the case $p = 1$.

Colouring H-Free Graphs

P_4

$P_3 + P_1$



Theorem (Král', Kratochvíl, Tuza & Woeginger, 2001)

Let H be a graph.

- *If H is an induced subgraph of P_4 or of $P_3 + P_1$, then* Colouring *is polynomial-time solvable for H-free graphs.*
- *If not, then* Colouring *is* NP*-complete for H-free graphs.*

Remark. Situation for (H_1, H_2)-free graphs is still *very* unclear (for example, the case $(K_{1,3}, P_r)$ is still open for $r \geq 6$).

3-Colouring for P_5-Free Graphs

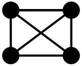
We consider the case $H = P_5$ (NP-complete case for COLOURING).

Let G be a P_5-free graph on n vertices.

Observation

If a graph G contains the K_4, then G is not 3-colourable.

Using brute force, we verify in $O(n^4)$ time if G contains K_4. If so, output `no`. Else we find that G is K_4-free.



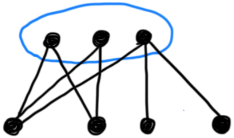
Observation

A graph G is 3-colourable if and only if every connected component of G is 3-colourable.

So, we may consider each connected component of G separately.

3-Colouring for P_5-Free Graphs

Now let G be a connected (K_4, P_5)-free graph.



Definition

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is dominating if every vertex outside D has at least one neighbour in D.

Bacsó and Tuza (1990): every connected P_5-free graph has a dominating set D that is either a clique (set of pairwise adjacent vertices) or induces a P_3.

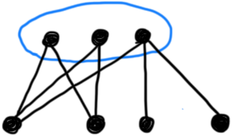
As G is not only connected and P_5-free but also K_4-free, we find:

Corollary

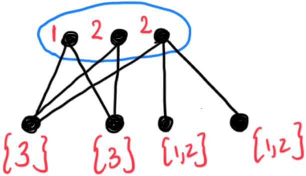
G has a dominating set D of size at most 3.

Using brute force, we find D in $O(n^3)$ time.

3-Colouring for P_5-Free Graphs



We now consider all options to colour the vertices of D.



For each option, we adjust the lists of possible colours of the vertices of $V - D$ accordingly.

This yields an instance of 2-List Colouring, a polynomial-time solvable problem (Edwards, 1986).

There are at most 3^3 options to colour the vertices of D. So we have at most 3^3 instances of 2-List Colouring to consider. ☐

k-Colouring P_r-Free Graphs

Theorem (Hoàng, Kamiński, Lozin, Sawada, Shu, 2010)

For every $k \geq 3$, k-COLOURING is polynomial-time solvable for P_5-free graphs.

Theorem (Bonomo, Chudnovsky, Maceli, Schaudt, Stein, Zhong, 2018)

3-COLOURING *is polynomial-time solvable for P_7-free graphs.*

Theorem (Pilipczuk, Pilipczuk, Rzążewski, 2021)

For every $r \geq 8$, 3-COLOURING is quasi-polynomial-time solvable for P_r-free graphs.

Theorem (Chudnovsky, Sprikl, Zhong, 2024)

4-COLOURING *is polynomial-time solvable for P_6-free graphs.*

Theorem (Huang, 2016)

5-COLOURING *is* NP*-complete for P_6-free graphs.*
4-COLOURING *is* NP*-complete for P_7-free graphs.*

k-Colouring H-Free Graphs

By using NP-completeness results of Emden-Weinert, Hougardy & Kreuter (1998), Holyer (1981) and Leven & Galil (1983) we obtain:

Theorem

Let H be a connected graph.

If $H \neq P_r$ for some $r \geq 1$, then k-Colouring for H-free graphs is NP*-complete for all $k \geq 3$.*

If $H = P_r$ for some $r \geq 1$, then

- 3-Colouring *for H-free graphs is polynomial-time solvable if $r \leq 7$ and quasi-polynomial-time solvable if $r \geq 8$.*
- 4-Colouring *for H-free graphs is polynomial-time solvable if $r \leq 6$ and* NP*-complete if $r \geq 7$.*
- 5-Colouring *for H-free graphs is polynomial-time solvable if $r \leq 5$ and* NP*-complete if $r \geq 6$.*
- Colouring *for H-free graphs is polynomial-time solvable if $r \leq 4$ and* NP*-complete if $r \geq 5$.*

Colouring *H*-Free Graphs

By using NP-completeness results of Emden-Weinert, Hougardy, Kreuter (1998), Holyer (1981) and Leven, Galil (1983) we obtain:

Theorem

Let H be a connected graph.

*If $H \neq P_r$ for some $r \geq 1$, then k-*Colouring *for H-free graphs is* NP*-complete for all $k \geq 3$.*

If $H = P_r$ for some $r \geq 1$, then

- 3-Colouring *for H-free graphs is polynomial-time solvable if $r \leq 7$ and quasi-polynomial-time solvable if $r \geq 8$.*
- 4-Colouring *for H-free graphs is polynomial-time solvable if $r \leq 6$ and* NP*-complete if $r \geq 7$.*
- k-Colouring ($k \geq 5$) *for H-free graphs is polynomial-time solvable if $r \leq 5$ and* NP*-complete if $r \geq 6$.*
- Colouring *for H-free graphs is polynomial-time solvable if $r \leq 4$ and* NP*-complete if $r \geq 5$.*

Open Problems

Open Problem

Can COLOURING *be solved in* $f(k)n^{O(1)}$ *time for* P_5*-free graphs?*

We can also go beyond hereditary graph classes. The diameter of a graph G is the maximum distance between two vertices of G.

Open Problem

Determine the complexity of 3-COLOURING *for graphs of diameter* 2*?*

Open Problem

Determine the complexity of 3-COLOURING *and* COLOURING *for triangle-free graphs of diameter* 2*?*

Open Problem

Does there exist an integer d *such that* COLOURING *is* NP*-complete for claw-free graphs of diameter* d*?*

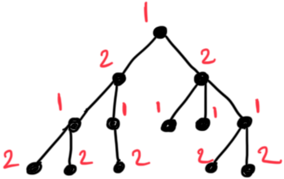
Colouring H-Subgraph-Free Graphs

Many NP-complete graph problems become polynomial-time solvable if the input is a tree (connected graph with no cycles).

Notion of treewidth measures how close a graph is to being a tree.

If a graph class $\mathcal{G}$ has small treewidth, then we can try to mimic efficient algorithms for trees to obtain efficient algorithms for $\mathcal{G}$.

Trees are 2-colourable.



A graph class $\mathcal{G}$ has bounded treewidth if there is a constant c such that every $G \in \mathcal{G}$ has treewidth at most c.

Theorem (Arnborg, Proskurowski, 1989)

Colouring *is linear-time solvable for every graph class of bounded treewidth.*

Colouring H-Subgraph-Free Graphs

a claw

a tripod
(subdivided claw)

Theorem (Robertson, Seymour, 1984)

For a connected graph H, the class of H-subgraph-free graphs has bounded treewidth if and only if H is a tripod or a path.

Corollary

If H is a tripod or a path, then Colouring *is linear-time solvable for H-subgraph-free graphs.*

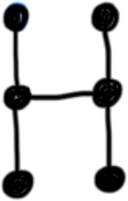
In contrast, for a connected graph H, Colouring on H-free graphs is only polynomial-time solvable if $H \subseteq P_4$ (if $P \neq \text{NP}$).

Colouring H-Subgraph-Free Graphs

The tripod is a tree of maximum degree 3 with exactly one vertex of degree 3.

There even exist trees H with two vertices of degree 3 for which COLOURING on H-subgraph-free graphs is polynomial-time solvable.

We will illustrate this by considering the "H"-graph $\mathbb{H}$.

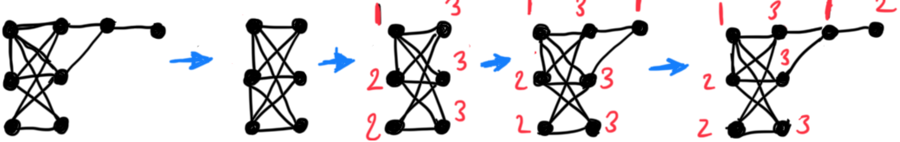


Colouring $\mathbb{H}$-Subgraph-Free Graphs

Let (G, k) be an instance of Colouring where G is $\mathbb{H}$-subgraph-free and $k \geq 3$.

We exhaustively delete vertices of degree at most 2. This takes polynomial time.

These vertices can always be coloured with a "free" colour when we restore them back into G in reverse order.

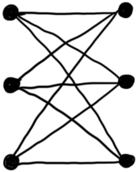


So after this process, every remaining vertex has at least three neighbours, that is, G has minimum degree at least 3.

Colouring $\mathbb{H}$-Subgraph-Free Graphs

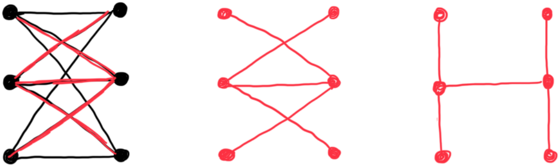
So G is $\mathbb{H}$-subgraph-free and has minimum degree at least 3.

$K_{3,3}$



Observation

Every $\mathbb{H}$-subgraph-free graph is also $K_{3,3}$-subgraph-free.



Colouring $\mathbb{H}$-Subgraph-Free Graphs

So, G is $\mathbb{H}$-subgraph-free and has minimum degree at least 3 and is also $K_{3,3}$-subgraph-free.

Theorem (Galvin, Rival, Sands, 1982)

There exists a constant d_ℓ such that every $K_{3,3}$-subgraph-free graph of treewidth at least d_ℓ has an induced P_ℓ.

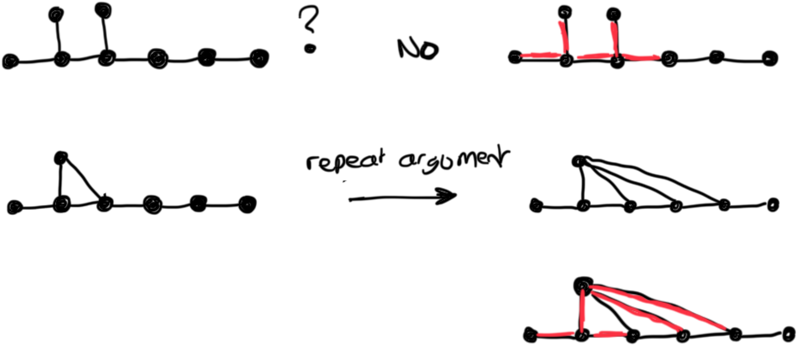
We claim that G has treewidth at most $d_6 - 1$, which means we can solve Colouring in polynomial time on G.

Else, by the above theorem, G contains an induced P_6.

Colouring $\mathbb{H}$-Subgraph-Free Graphs

So, G is $\mathbb{H}$-subgraph-free, has minimum degree at least 3 and contains an induced P_6.

The inner vertices of the P_6 all need a third neighbour.



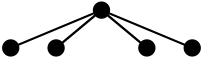
So G must have indeed treewidth at most d_6. □

Colouring H-Subgraph-Free Graphs

So there exist trees H with two vertices of degree 3 for which COLOURING on H-subgraph-free graphs is polynomial-time solvable.

We now consider trees H with a vertex of degree 4.

Let $H = K_{1,4}$.



Observation

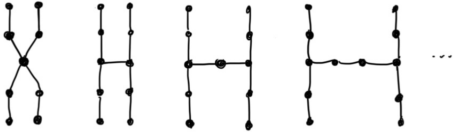
A graph G is $K_{1,4}$-subgraph-free if and only if G has maximum degree at most 3.

Hence, by Brooks' Theorem (1941), every $K_{1,4}$-subgraph-free graph G can be coloured with at most 3 colours unless $G = K_4$.

Colouring H-Subgraph-Free Graphs

Let H be a subdivided $K_{1,4}$ or a subdivided $\mathbb{H}$.

We say that H is difficult if it contains one of:



Else we say that H is easy.

Theorem (Golovach, P., Ries, 2012)

For every difficult H, 3-Colouring is NP*-complete for H-subgraph-free graphs.*

Extending results of Golovach, P. & Ries (2012) and Johnson, Martin, Pandey, P., Smith & van Leeuwen (2023):

Theorem (Eagling-Vose, Jooken, Lucke, Martin, P., 2026)

For every easy H, 3-Colouring is polynomial-time solvable for H-subgraph-free graphs.

Colouring H-Subgraph-Free Graphs

We now consider the case where H is a tree that contains a vertex of degree 5.

Theorem (Garey, Johnson, Stockmeyer, 1977)

3-Colouring *is* NP*-complete for* $K_{1,5}$*-subgraph-free graphs.*

Finally, we consider the case where H contains a cycle.

Let C_s denote the cycle on s vertices (so C_3 is the triangle).

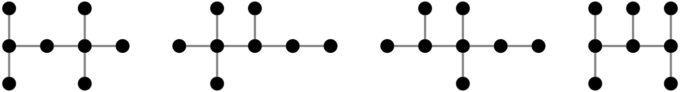
Theorem (Emden-Weinert, Hougardy, Kreuter, 1998)

For all $s \geq 3$*,* 3-Colouring *is* NP*-complete for* C_s*-subgraph-free graphs.*

Colouring *H*-Subgraph-Free Graphs

The complexity of Colouring on H-subgraph-free graphs has now been settled for all connected graphs H except when H is

- a tree of maximum degree 4 with exactly one vertex of degree 4 and at least one vertex of degree 3; or
- a tree of maximum degree 3 with at least three vertices of degree 3.



The four open cases of graphs H on eight vertices.

Future Directions: Classifications

Complete the dichotomies for

- 3-COLOURING for P_t-free graphs when $t \geq 8$
- k-COLOURING ($k \geq 3$) for H-free graphs and disconnected H
- COLOURING for (H_1, H_2)-free graphs
- COLOURING for H-subgraph-free graphs

The aim is that in this way new techniques are developed that are also useful for other graph problems.

Wider Applicability

For example, graph problems that are

C1. polynomial-time solvable for graphs of bounded treewidth;
C2. NP-complete for subcubic graphs (i.e. of max degree ≤ 3);
C3. stay NP-complete under edge subdivision of subcubic graphs.

are fully classified for $\mathcal{H}$-subgraph-free graphs for all finite sets $\mathcal{H}$. See the survey of Johnson, Martin, Oostveen, Pandey, P., Smith, van Leeuwen (2025) for a list of 20+ C123-problems.

Colouring does not satisfy C2 (due to Brooks' Theorem).

Our graph decomposition theorems for Colouring for H-subgraph-free graphs can also be applied to some extent to other classic graph problems that do not satisfy C2, such as:

- Stable Cut
- Feedback Vertex Set
- Connected Vertex Cover
- Matching Cut

.

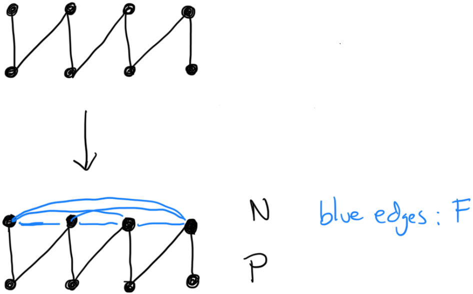
Probe Graphs

Let H be a graph. A partitioned probe H-free graph (G, P, N) consists of

- a graph $G = (V, E)$,
- a set $P \subseteq V$ of probes, and
- an independent set $N = V \setminus P$ of non-probes,

such that $G + F$ is H-free for some edge set $F \subseteq \binom{N}{2}$.

Example.
Every P_r is probe P_5-free.



Probe Graphs

We recall:

Theorem (Hoàng, Kamiński, Lozin, Sawada, Shu, 2010)

For every $k \geq 3$, k-COLOURING is polynomial-time solvable for P_5-free graphs.

Theorem (P., Rauch, van Leeuwen, 2026)

3-COLOURING is polynomial-time solvable for partitioned probe P_5-free graphs

Open Problem

For $k \geq 4$, determine the complexity of k-COLOURING for partitioned probe P_5-free graphs.

Final Open Problems

Open Problem

Can probe P_5-free graphs be recognized in polynomial time?

Theorem (P., Rauch, van Leeuwen, 2026)

Probe P_5-free graphs are a proper subclass of $(C_7, C_9, C_{11}, \ldots)$-free graphs.

Ochem (graphclasses.org) observed that 3-Colouring can be solved in polynomial time for odd-hole-free graphs, that is, $(C_5, C_7, C_9, \ldots)$-free graphs.

Open Problem

What is the complexity of 3-Colouring on $(C_7, C_9, C_{11}, \ldots)$-free graphs?