k-Sum Hardness Implies tw-SETH

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Main result

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Theorem

If there exists a $(2-\varepsilon)^{tw} n^{O(1)}$ *time algorithm for* 3-SAT
$\Rightarrow$ *there exists an* $n^{(1-\varepsilon)\frac{k}{2}}$ *algorithm for* k-SUM.

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[a] Randomized, for some other ε and sufficiently large k

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Image credit: nano banana

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- Same statements hold for k-XOR.
- Result presented in SODA 2026.

Fair Warning

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Talk will be heavy on:

- Controversial Opinions
- **Propaganda**

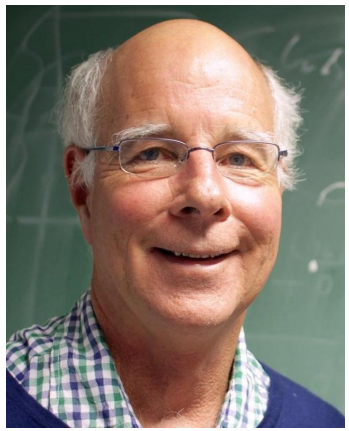
Objective:

- Build case for research direction
- (**Entertain**)

Background

Parameterized Complexity

- Do **not** measure complexity as function of input **size** n.
- Measure complexity as function of input **structure** k.
- k is the **parameter**.
- Motivating difference: $f(k)n^{O(1)}$ vs $n^{g(k)}$.



Engineer's Motivation:

- If $f(k)$ is reasonable, this could actually be a useful algorithm!

Scientist's Motivation:

- **Why** are problems hard?

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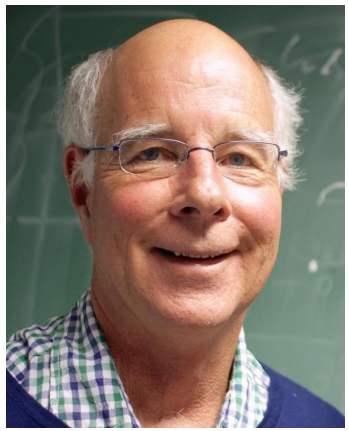

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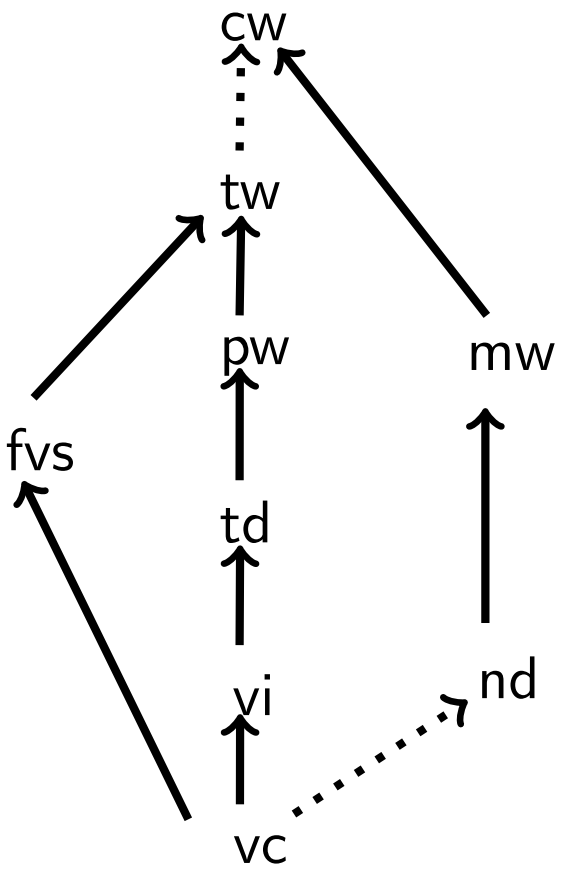
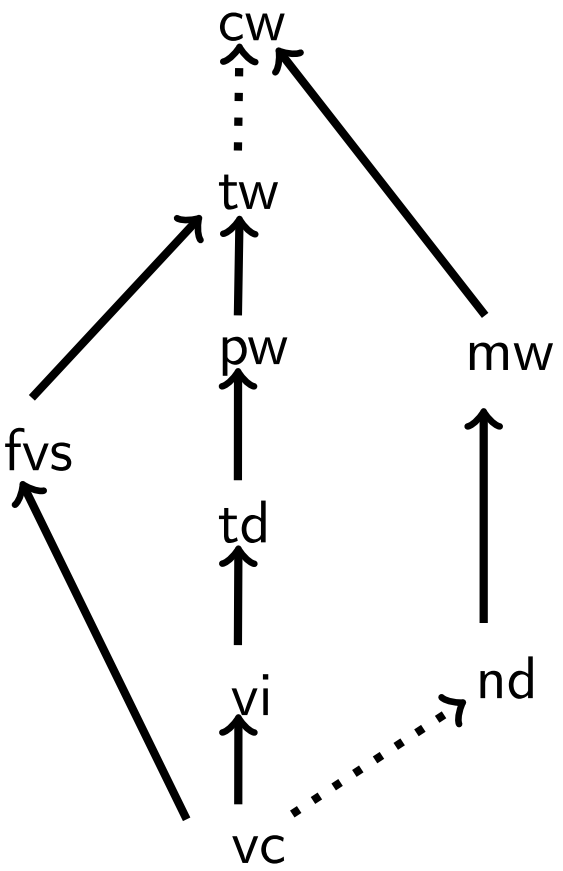
Bottom line: Complexity must take into account **input structure**.

Which parameters – which structure?



Which parameters – which structure?

- Key parameter : **Treewidth**
  - $\mathrm{tw}(G)$ bounded $\rightarrow$ G is "tree-like"



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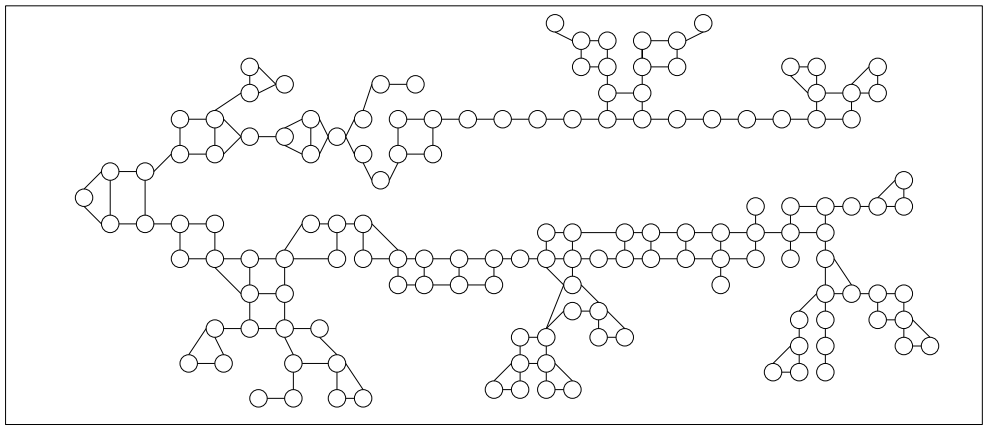

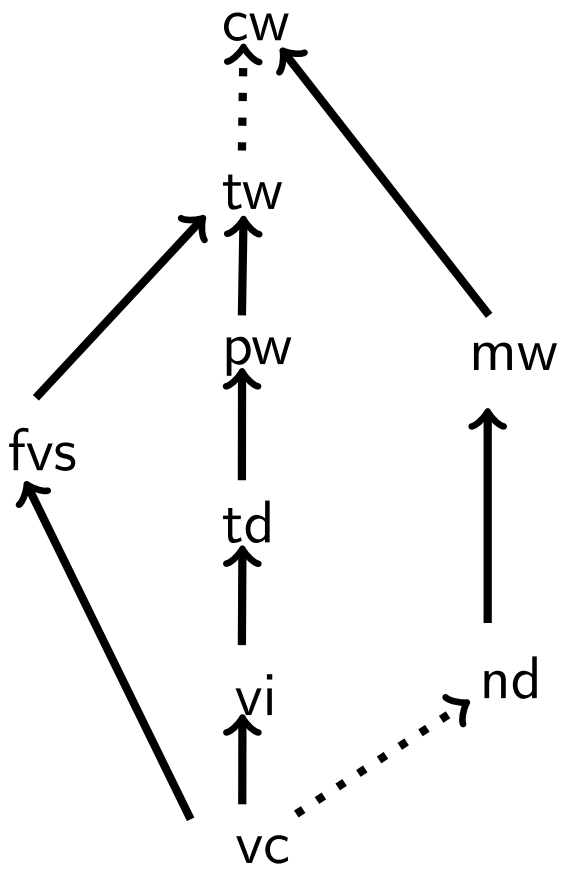

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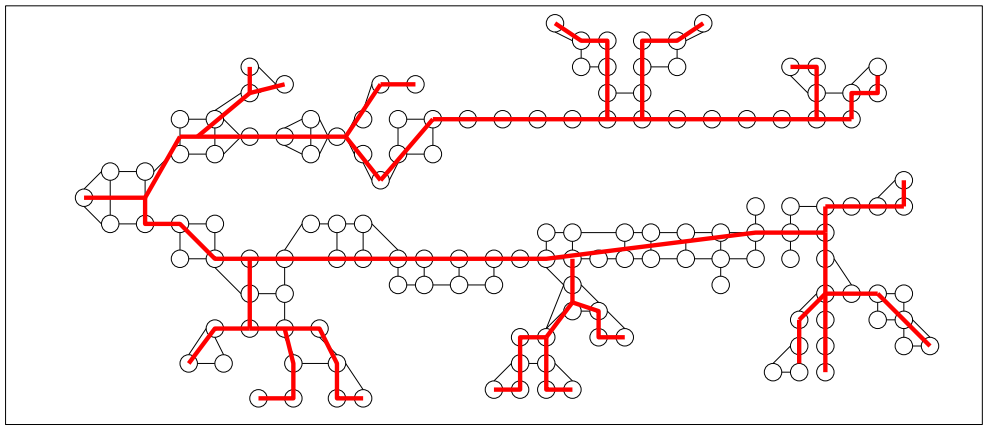

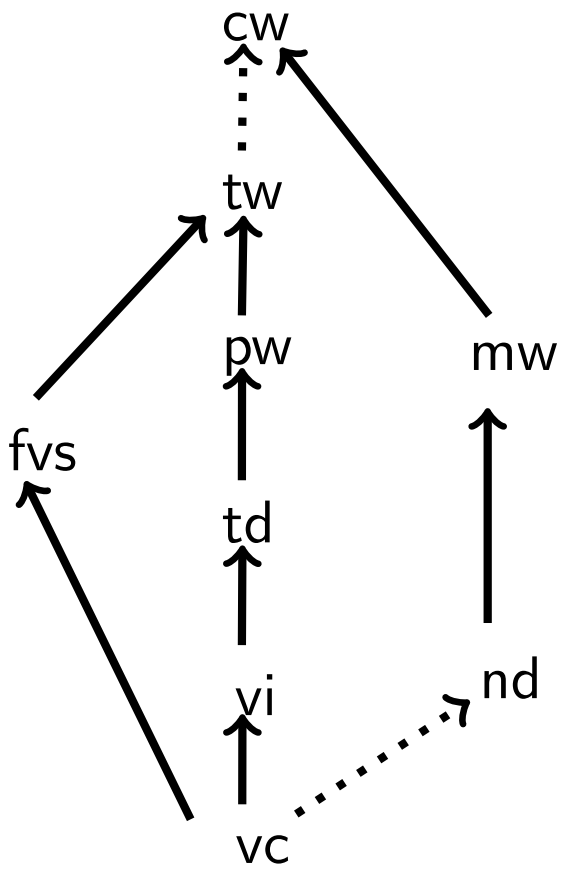

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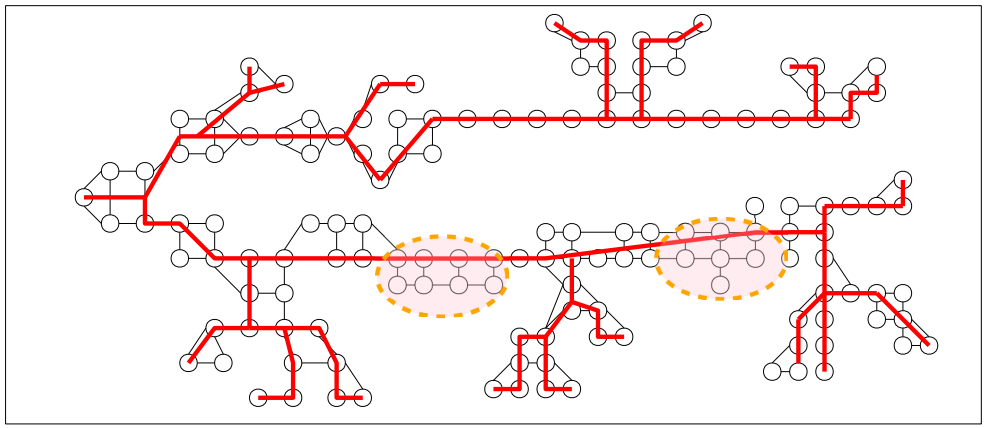

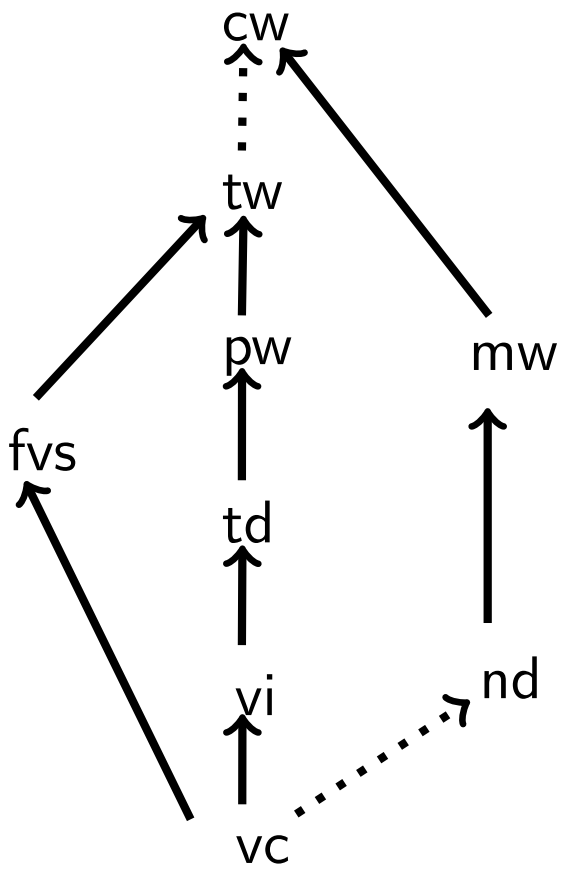

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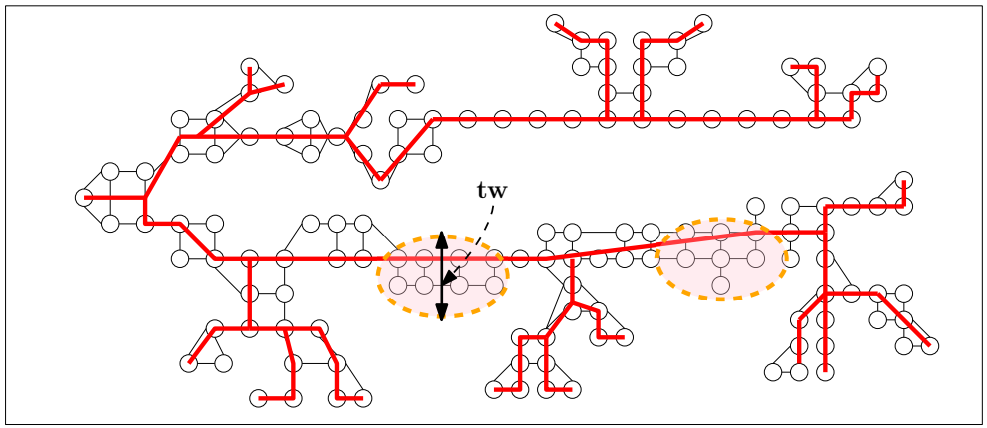

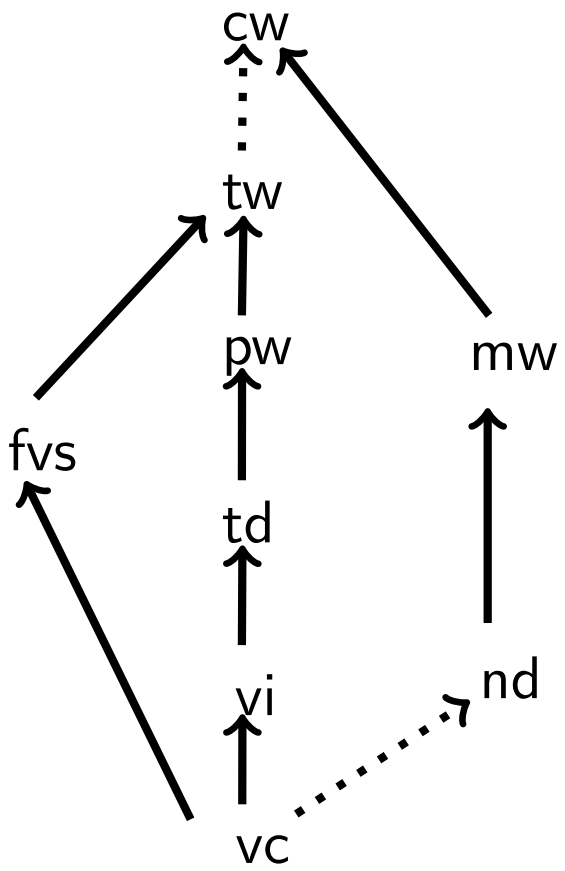

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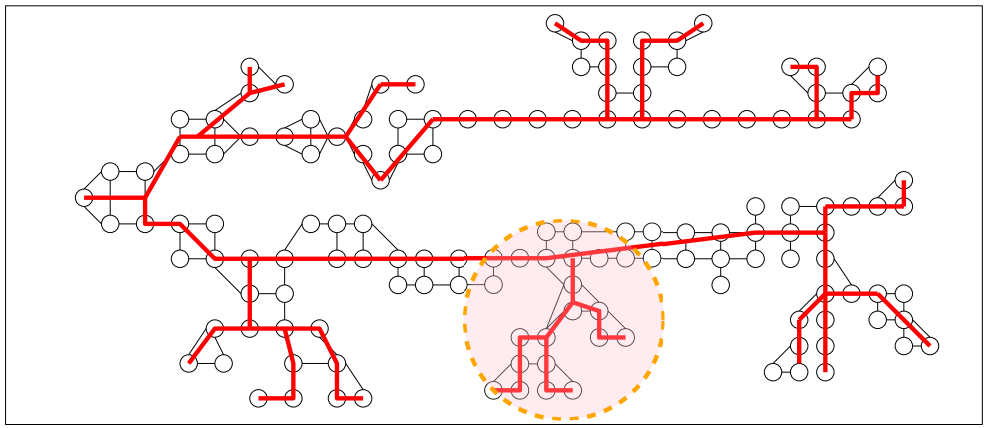

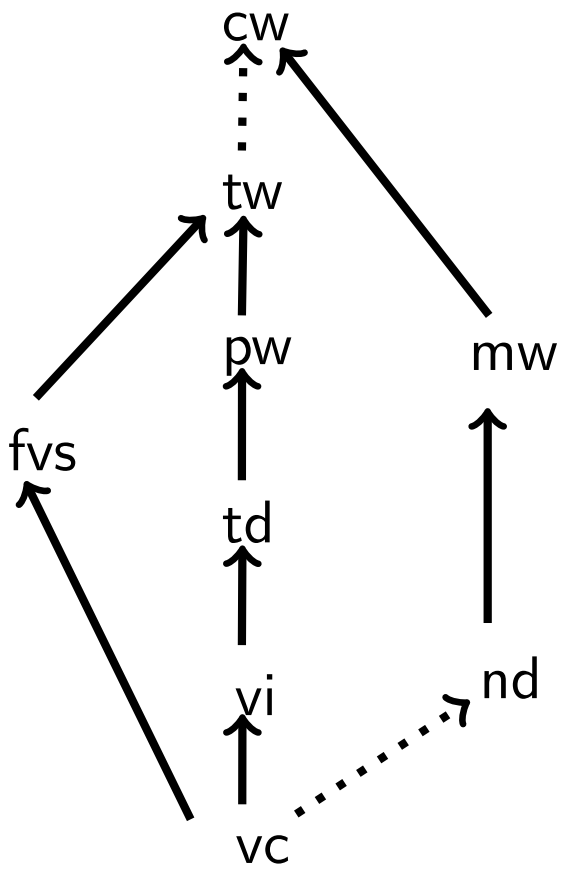

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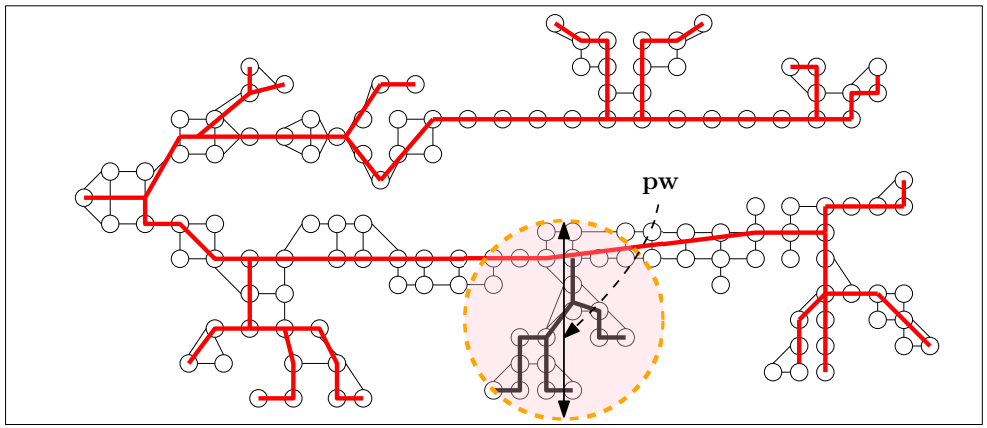

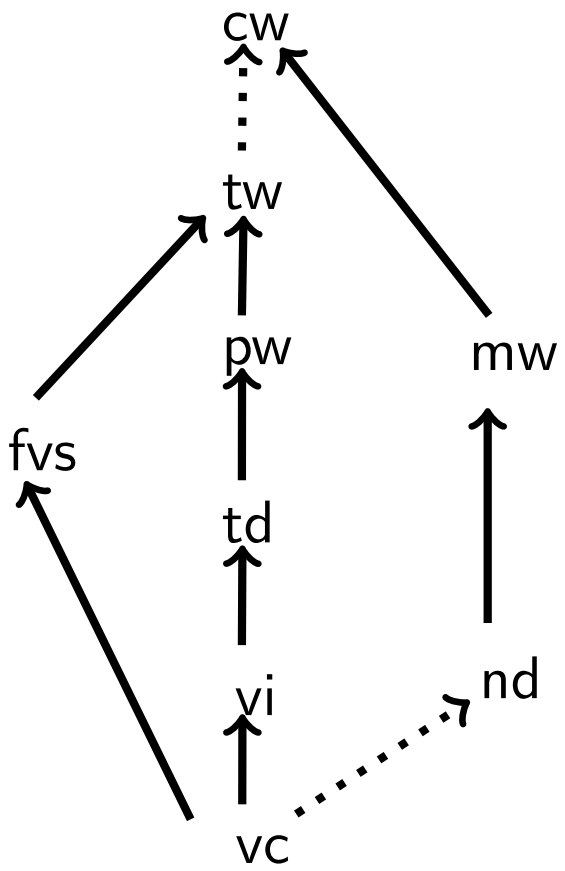

Concretely :

- Pathwidth should be **easier** than treewidth !
  - For which problems ?
  - Complexity impact ? $\rightarrow$ **Fine-Grained complexity**

Treewidth – Pathwidth

Gentle definition of pathwidth k:

- We have k stacks. Initially each contains a vertex. They are arbitrarily connected.
- At each step we add a vertex to the top of a stack. It can be connected to vertices currently on top of a stack.

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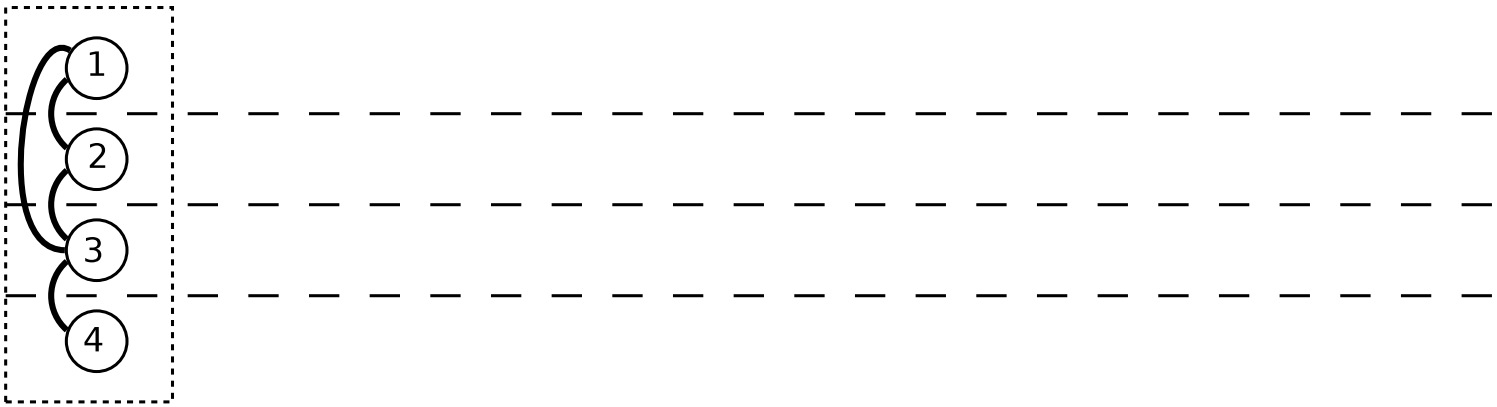

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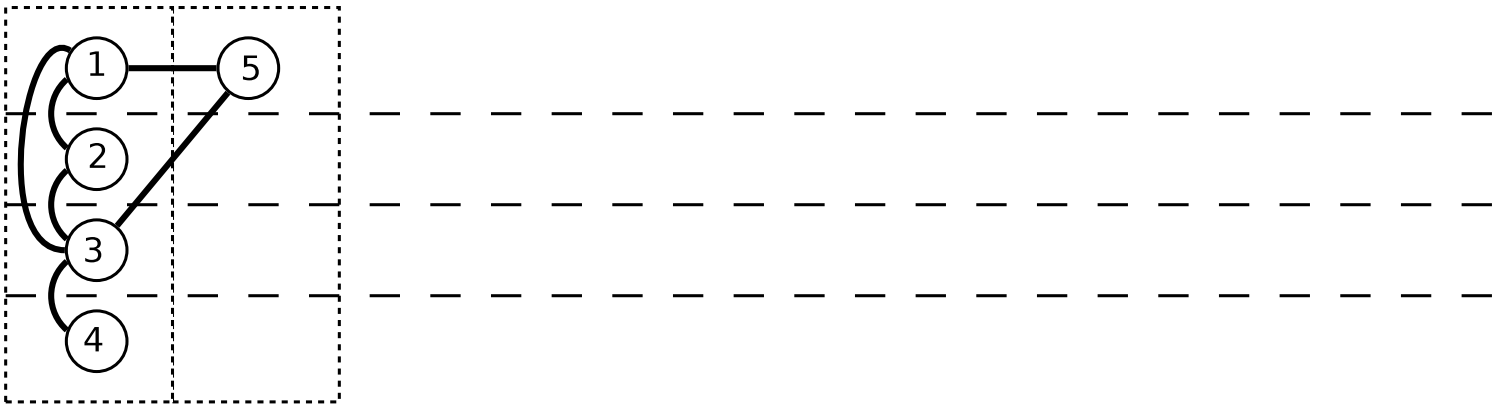

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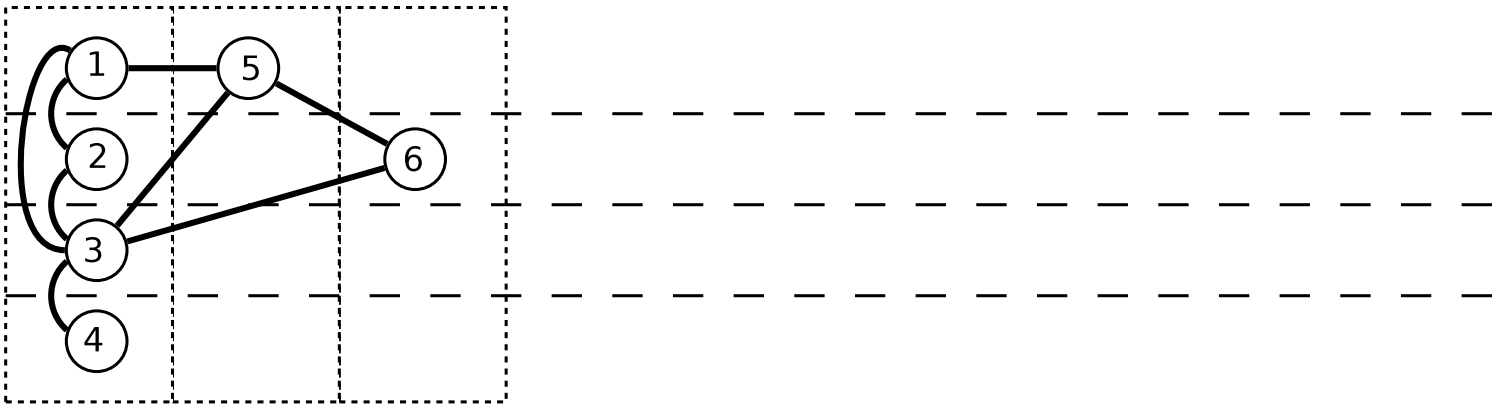

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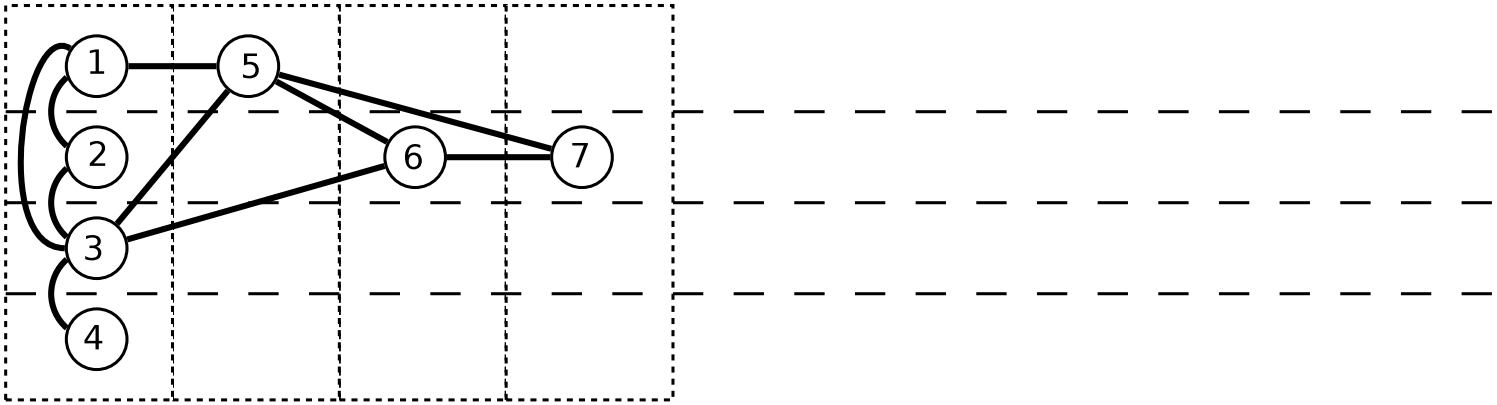

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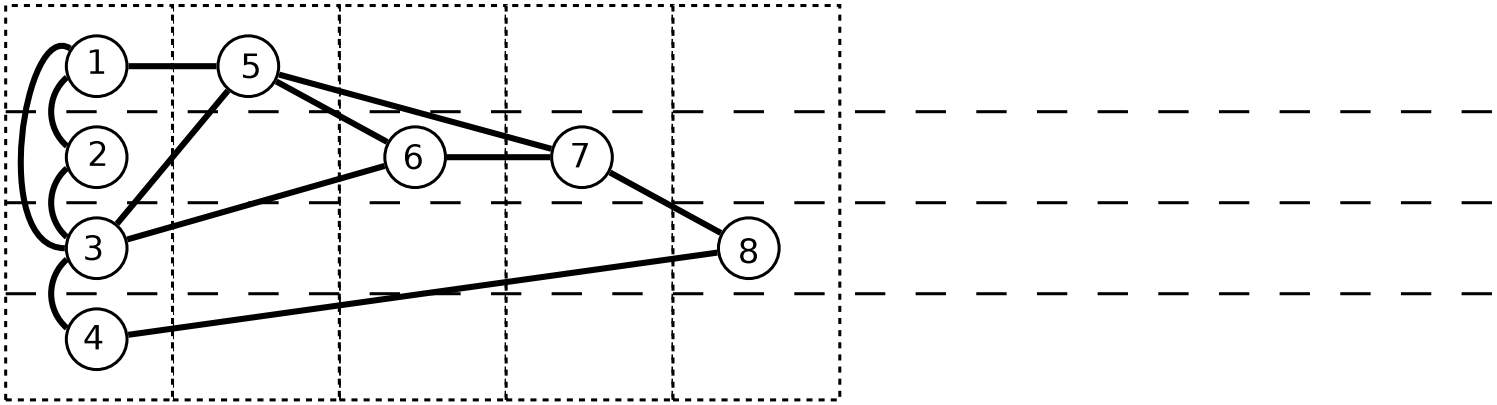

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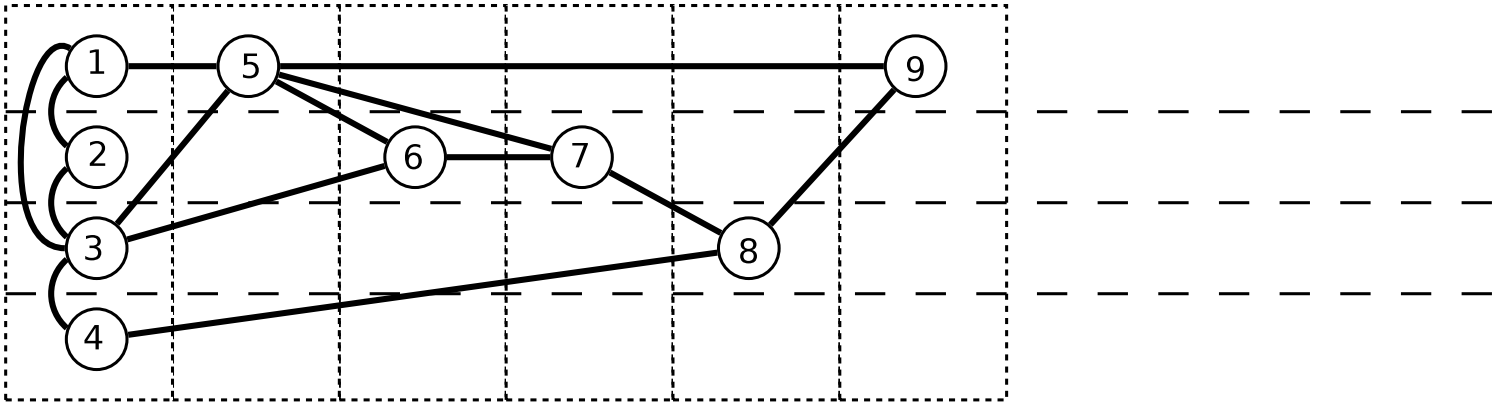

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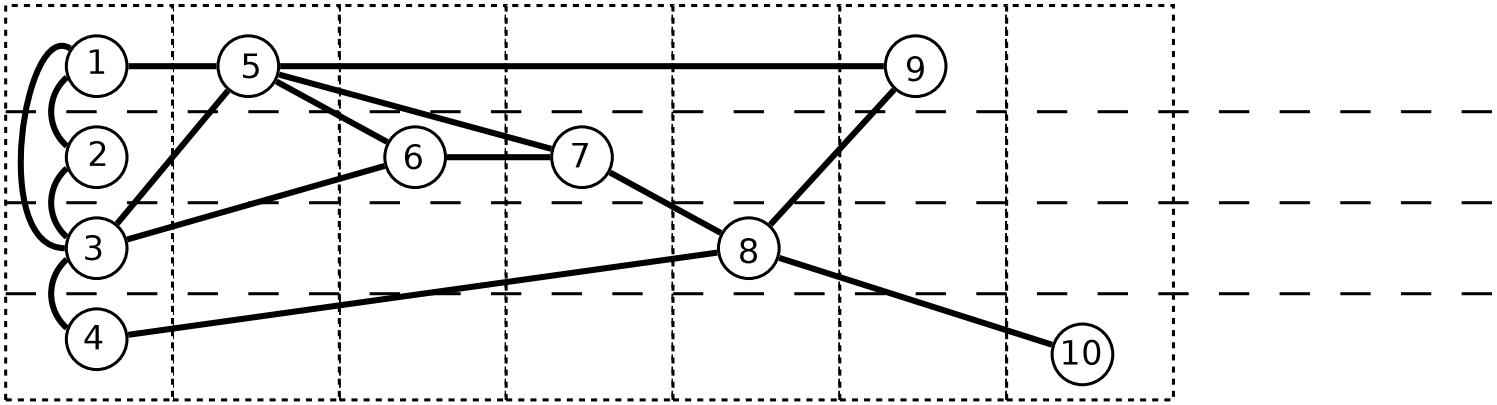

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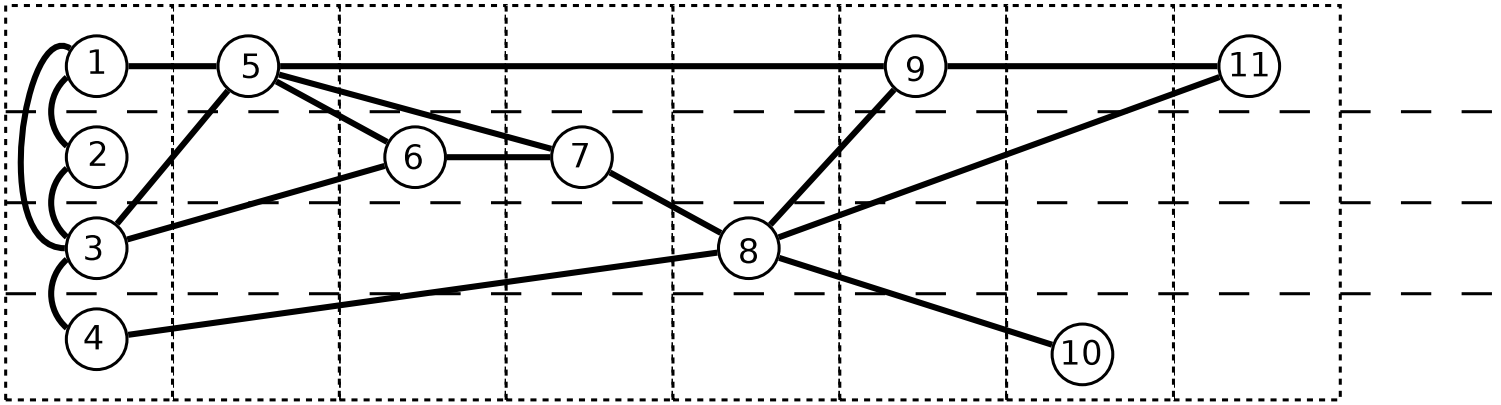

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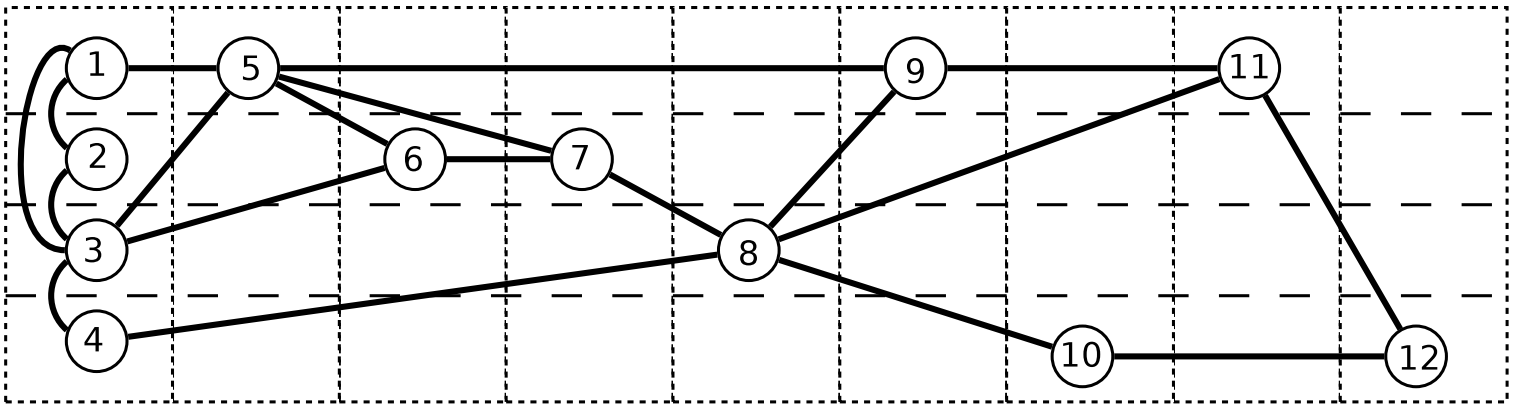
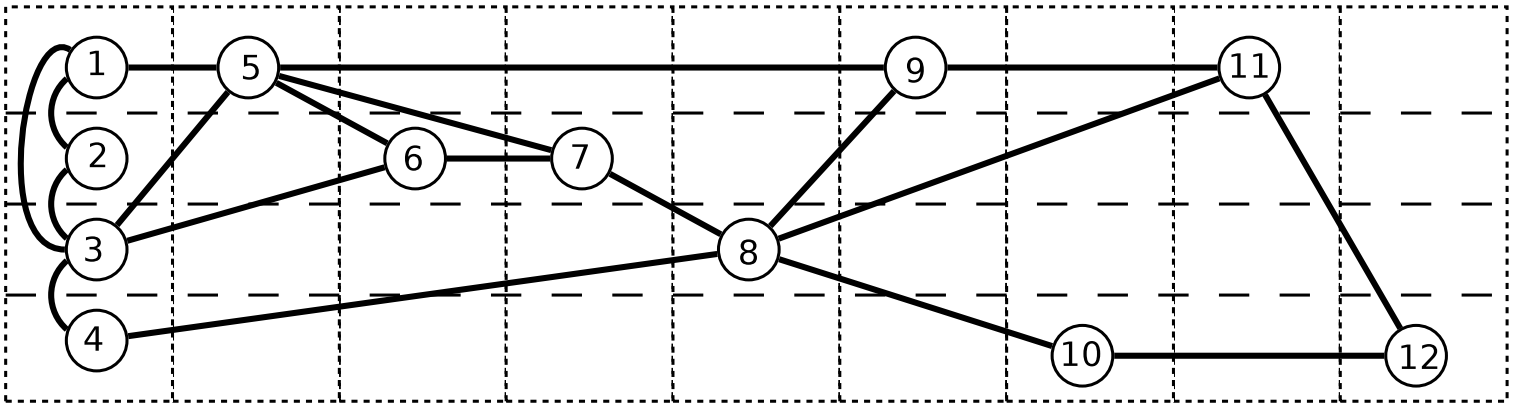


Treewidth – Pathwidth

Note that this is equivalent to the standard definition of path decompositions.



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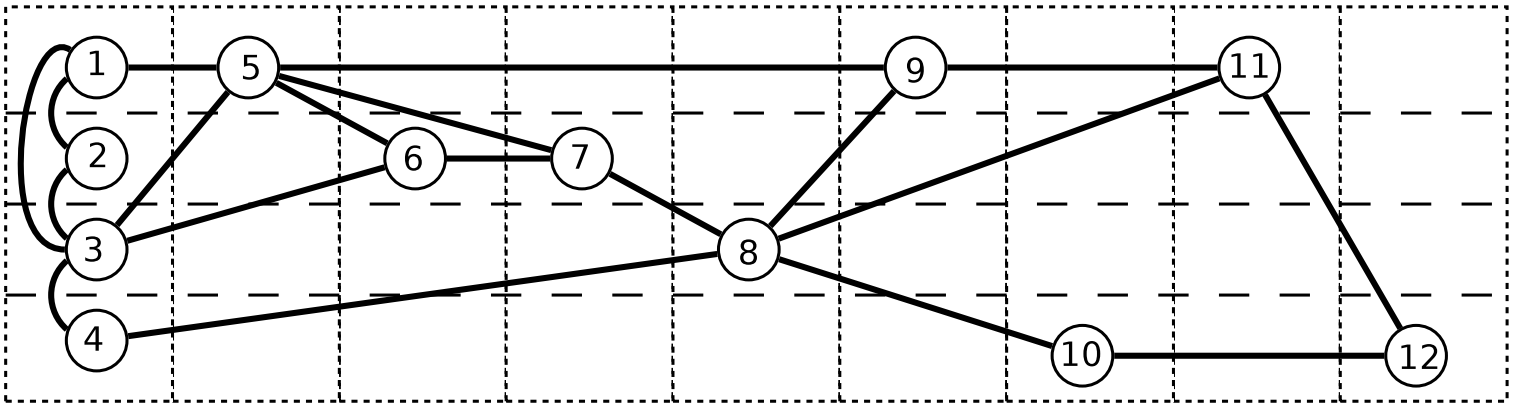

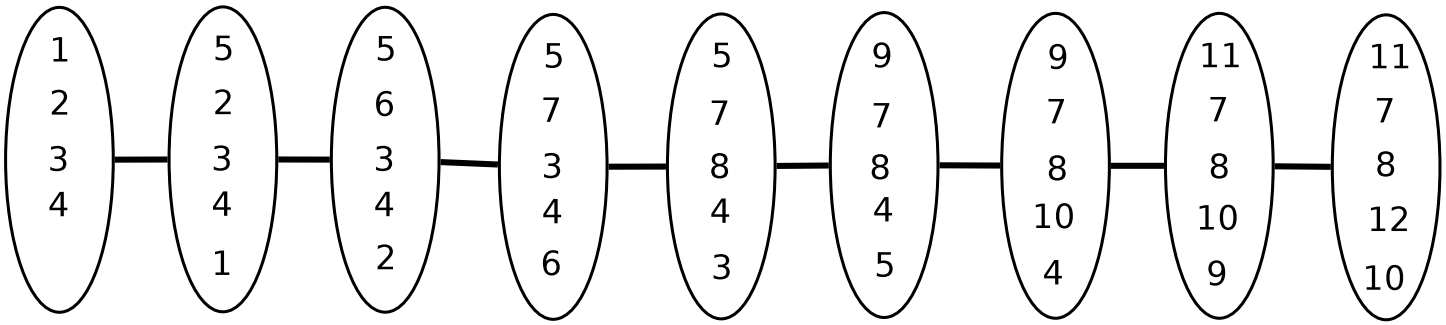

Algorithmic view

The reason that tree/path decompositions are useful is that we have a moving boundary of small separators that "sweeps" the graph.

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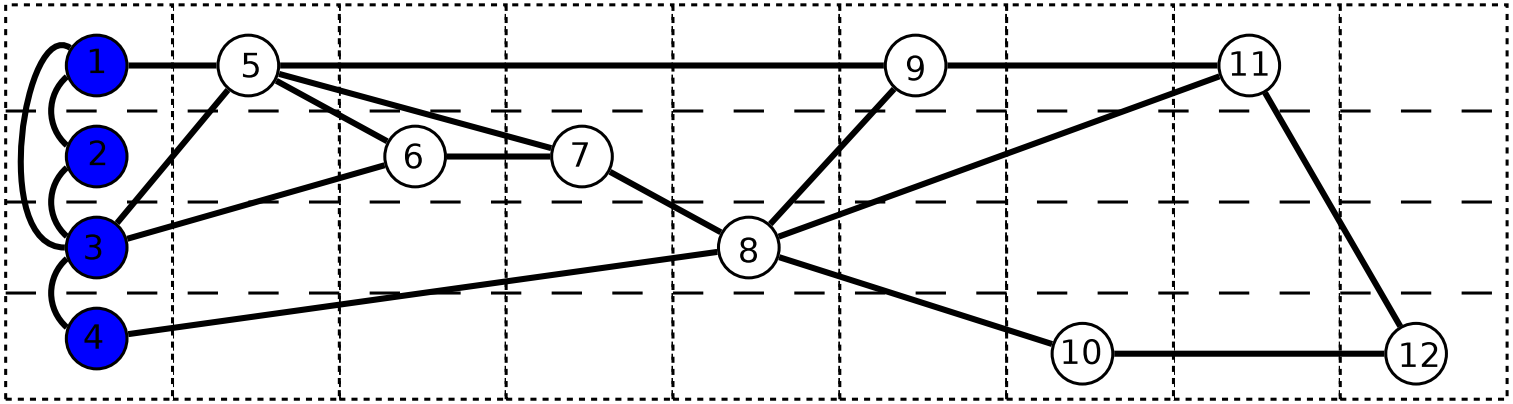

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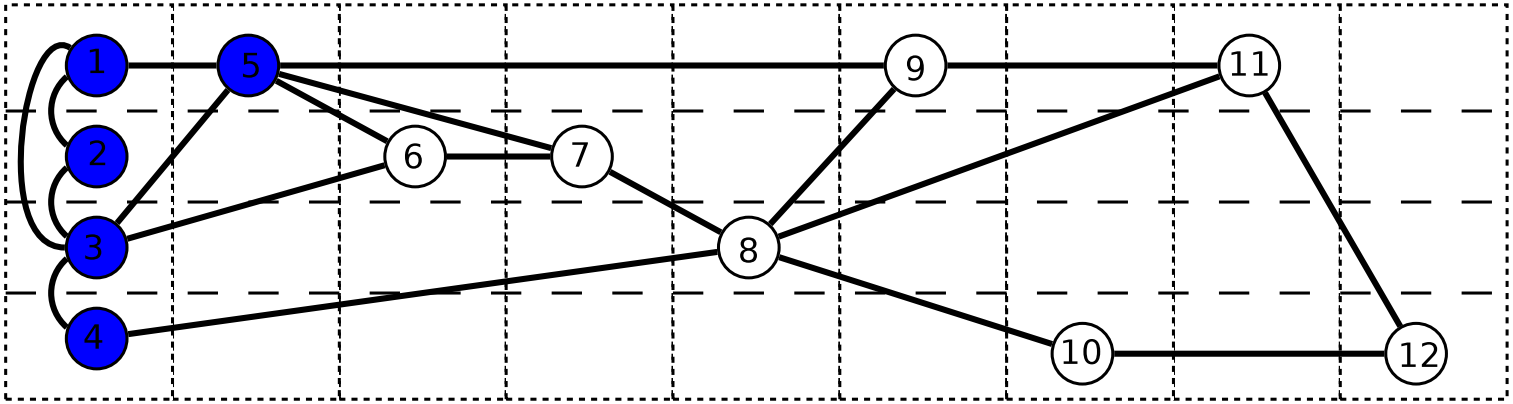

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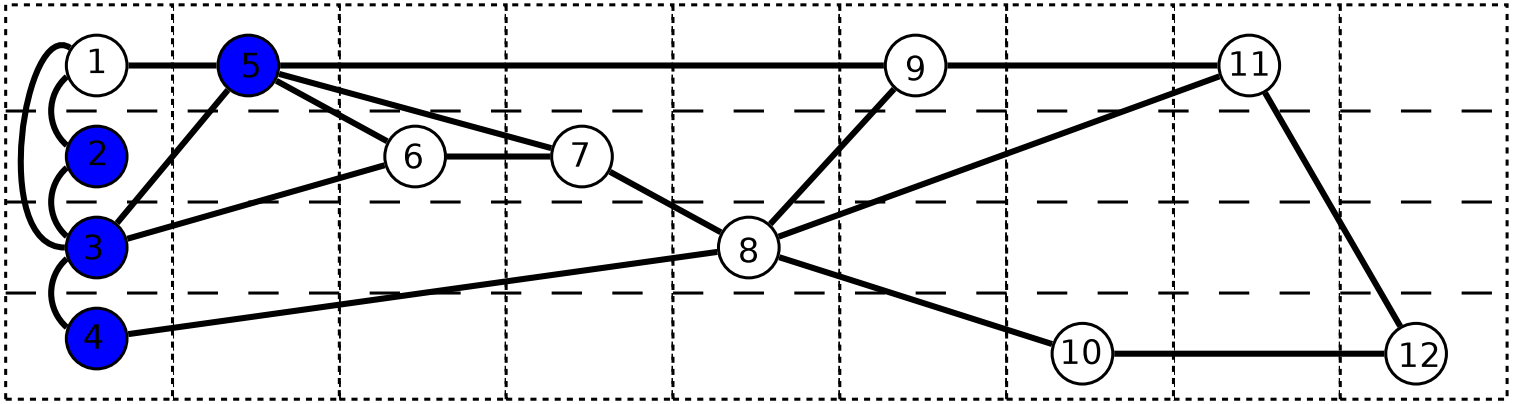

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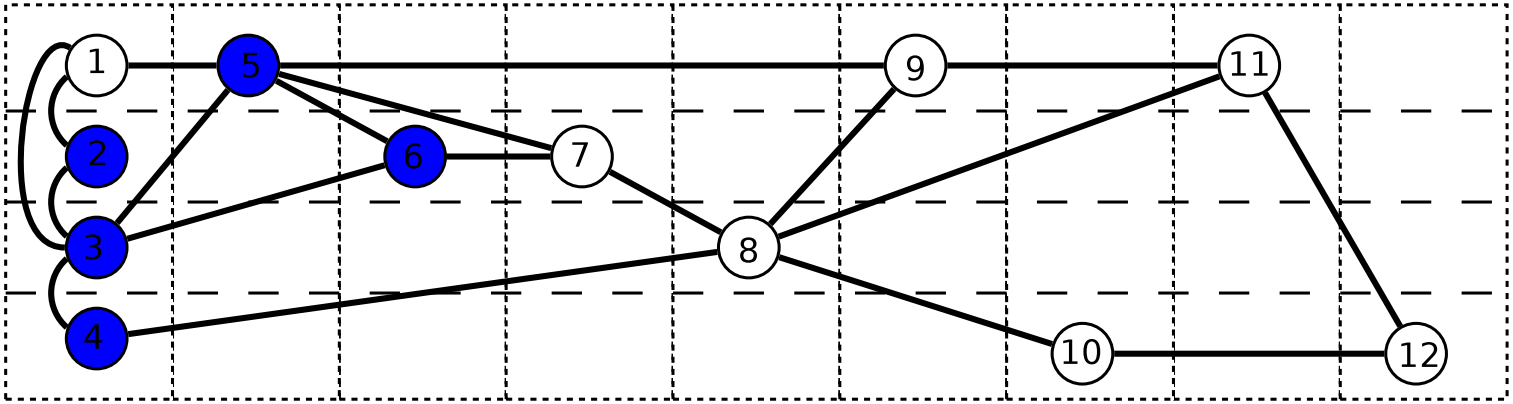

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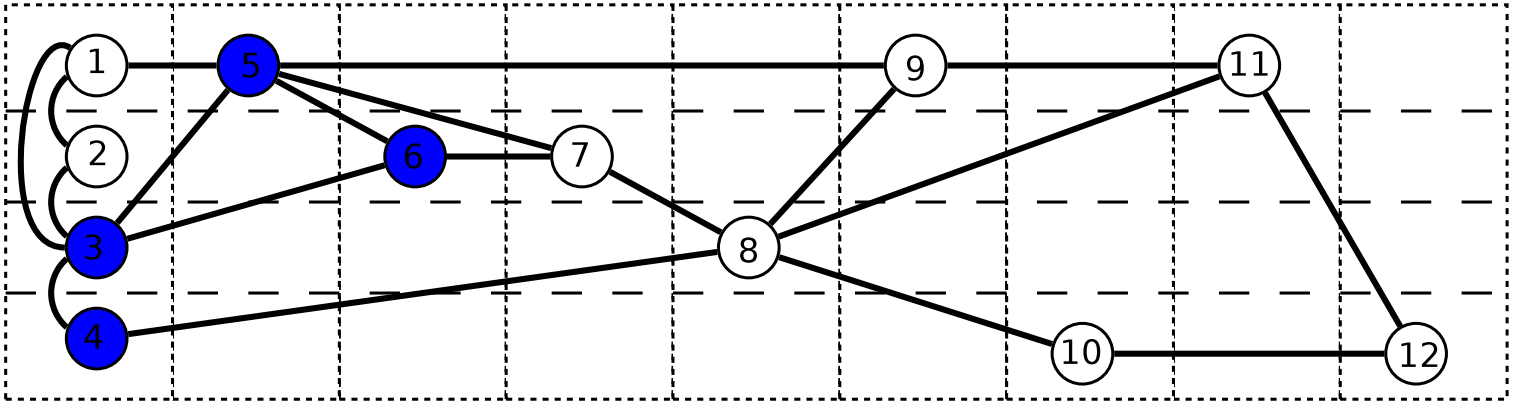

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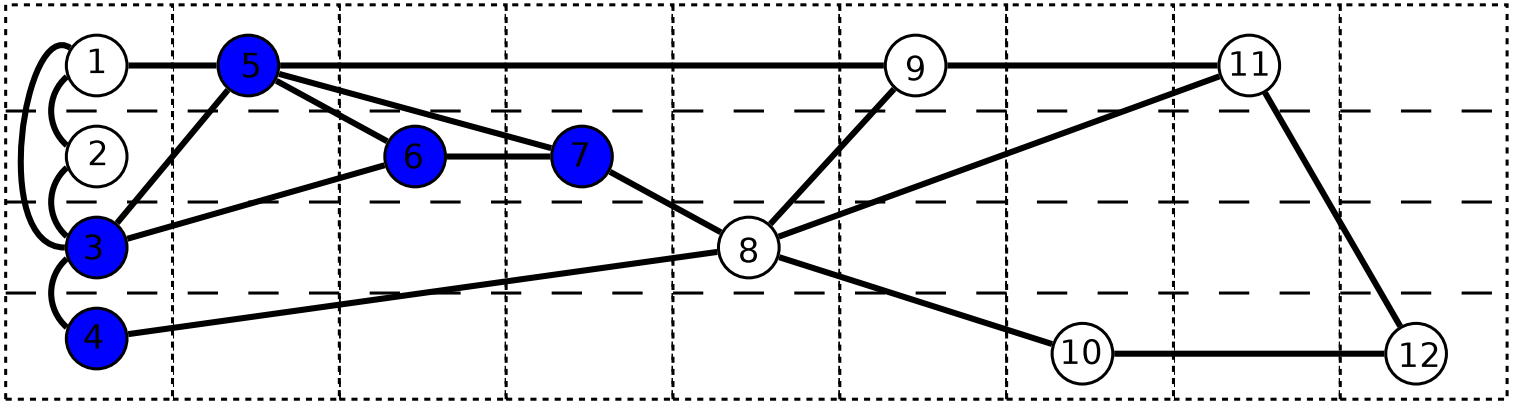

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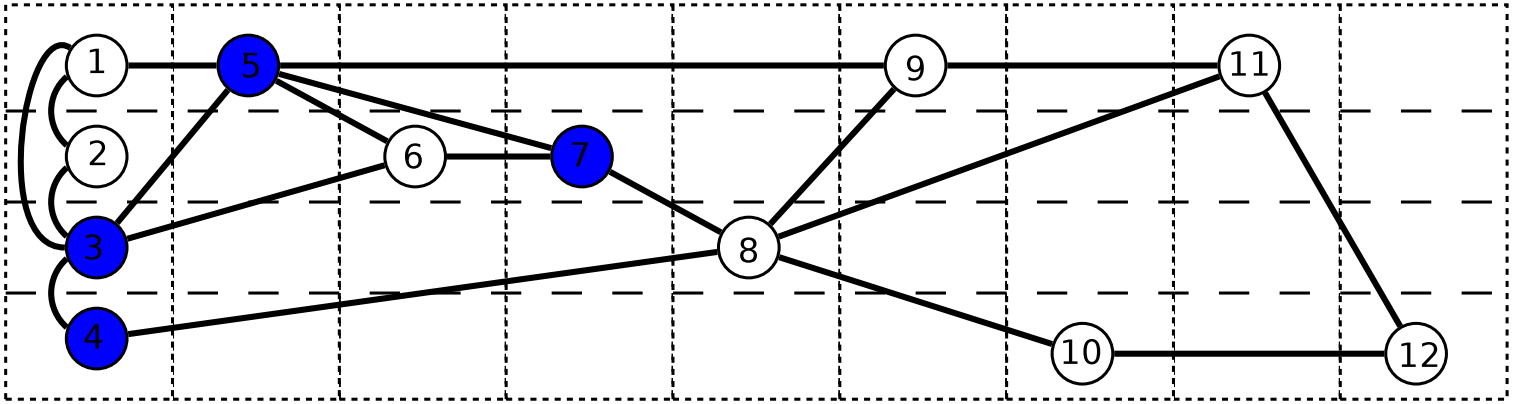

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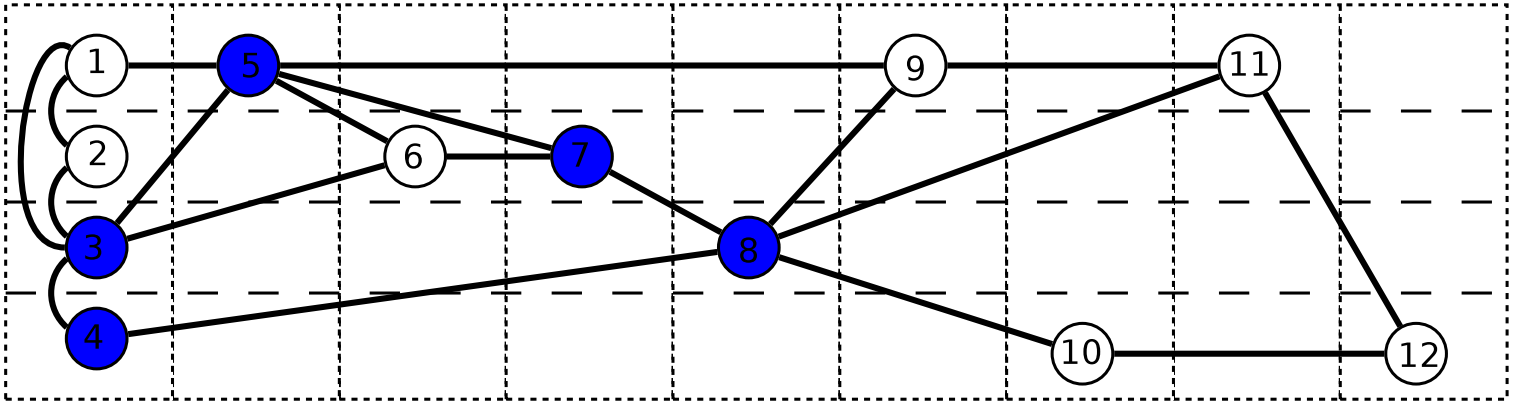

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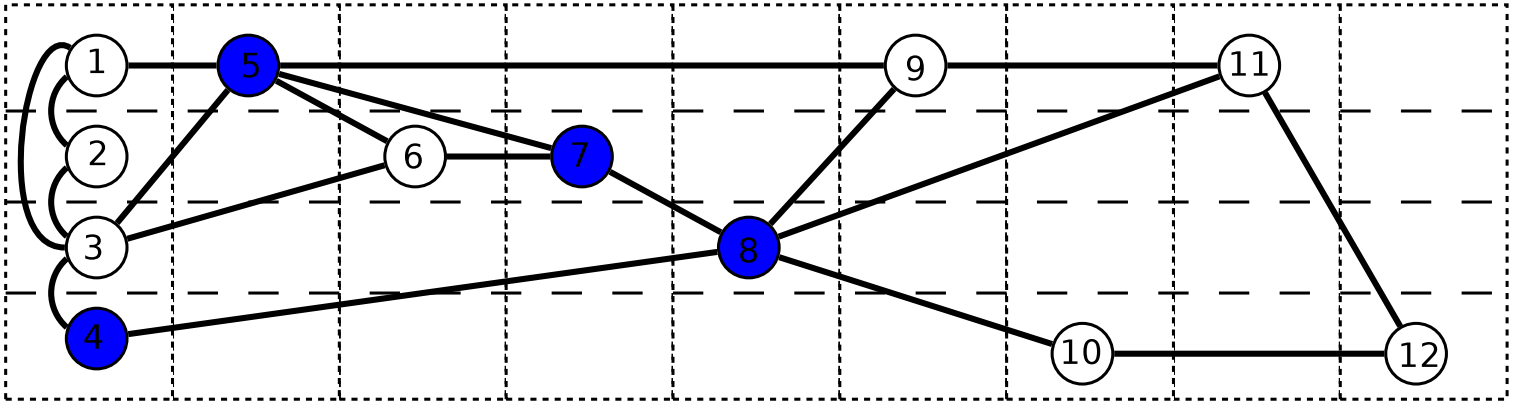

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Which colorings of the boundary are properly extendible to the left?

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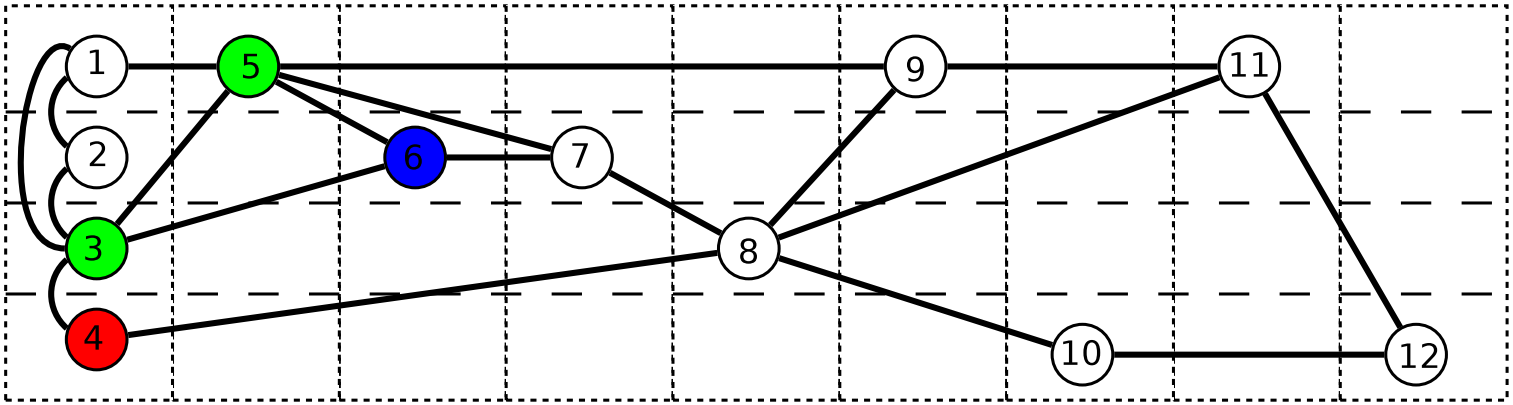

Separator: $\{3, 4, 5, 6\}$ includes tuple (3,4,5,6;No) because this coloring does not work

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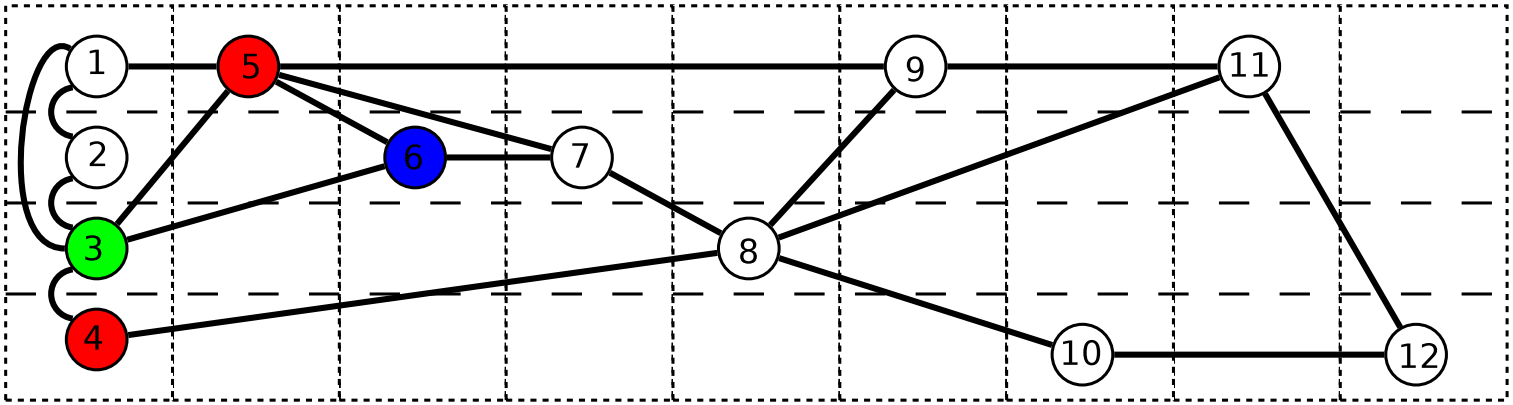

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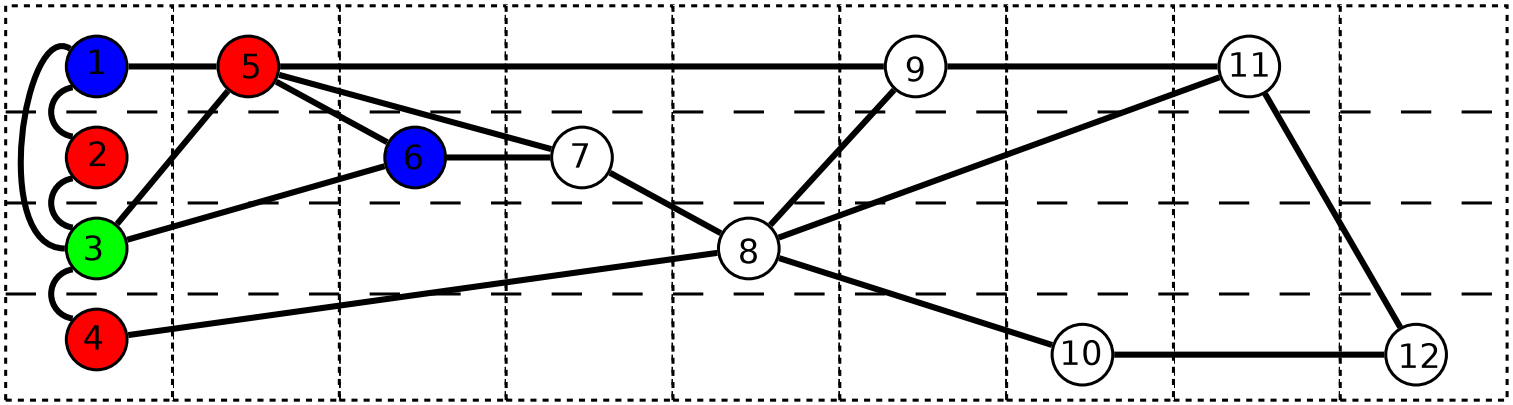

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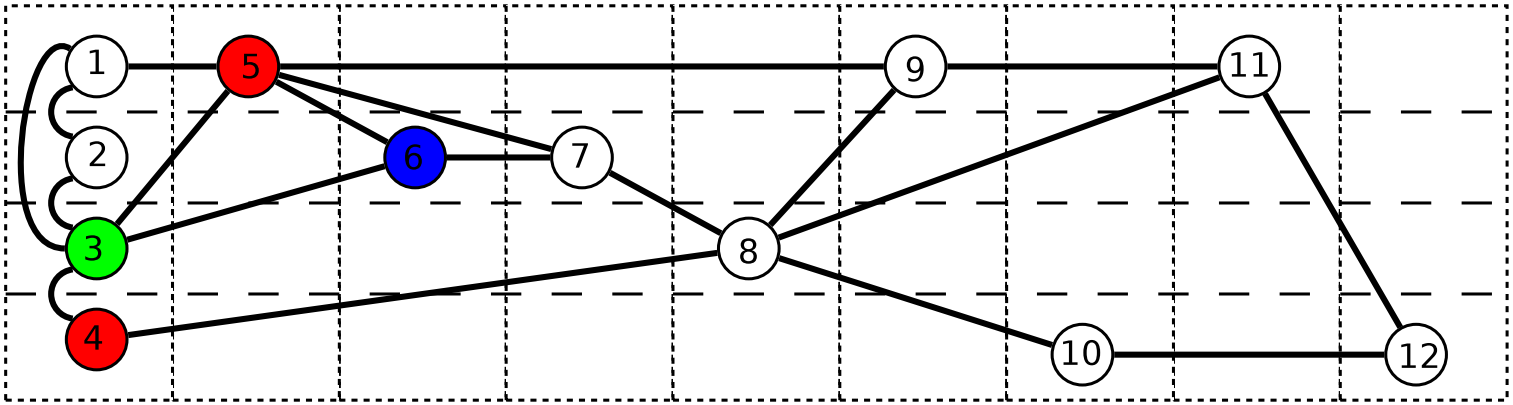

We now need to decide which are the good colorings for the separator $(3, 4, 5, 7)$.

We consider each good coloring of $(3, 4, 5, 6)$.

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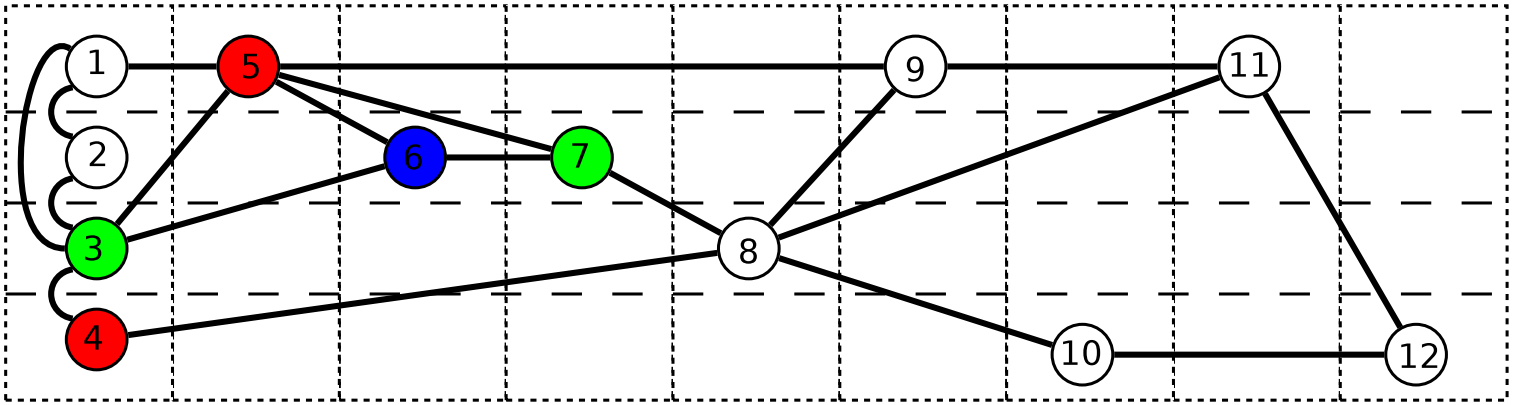

We now need to decide which are the good colorings for the separator $(3, 4, 5, 7)$.

We consider each good coloring of $(3, 4, 5, 6)$.

We see that $(3, 4, 5, 7)$ is a good coloring.

Important: we know the colors of all neighbors of 7.

Application: Treewidth Lower Bounds

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Image credit: nano banana (including typo!)

Application: Treewidth Lower Bounds

	Pathwidth	Treewidth	Cliquewidth	Ref
Ind Set	$= 2^w$	$= 2^w$	$= 2^w$	Lokshtanov,Marx,Saurabh TALG'18
Dom Set	$= 3^w$	$= 3^w$	$= 4^w$	LMS'18 and Katsikarelis, L., Paschos DAM'19
k-Col	$= k^w$	$= k^w$	$= (2^k - 2)^w$	LMS'18 and L. SIDMA'20
Max Cut	$= 2^w$	$= 2^w$	W[1]-h	LMS'18 and Fomin, Golovach, Lokshtanov, Saurabh SICOMP'10
Conn VC	$= 3^w$	$= 3^w$	$= 6^w$	Cygan et al. TALG'22, Hegerfeld, Kratsch ESA'23
Ham Cycle	$= (2+\sqrt{2})^w$	$\leq 4^w$	W[1]-h	FGLS'18, Cygan, Kratsch, Nederlof JACM'18

All lower bounds known under SETH.

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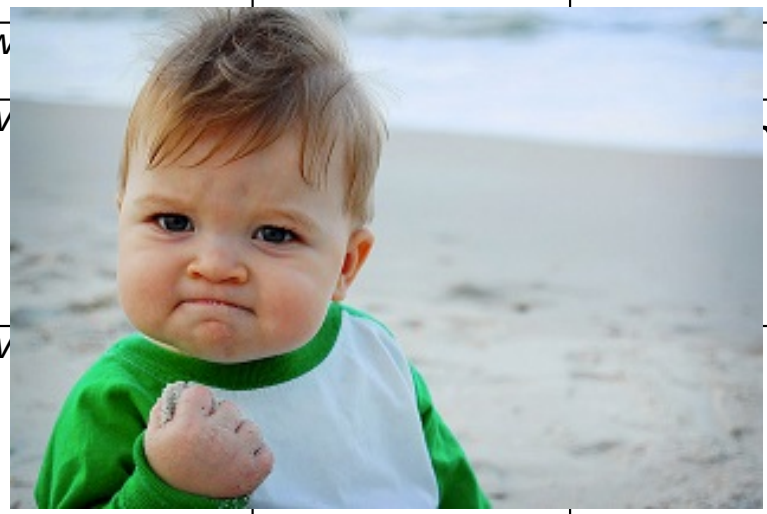

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All lower bounds known under SETH.

Motivation and Context

Motivation and Context

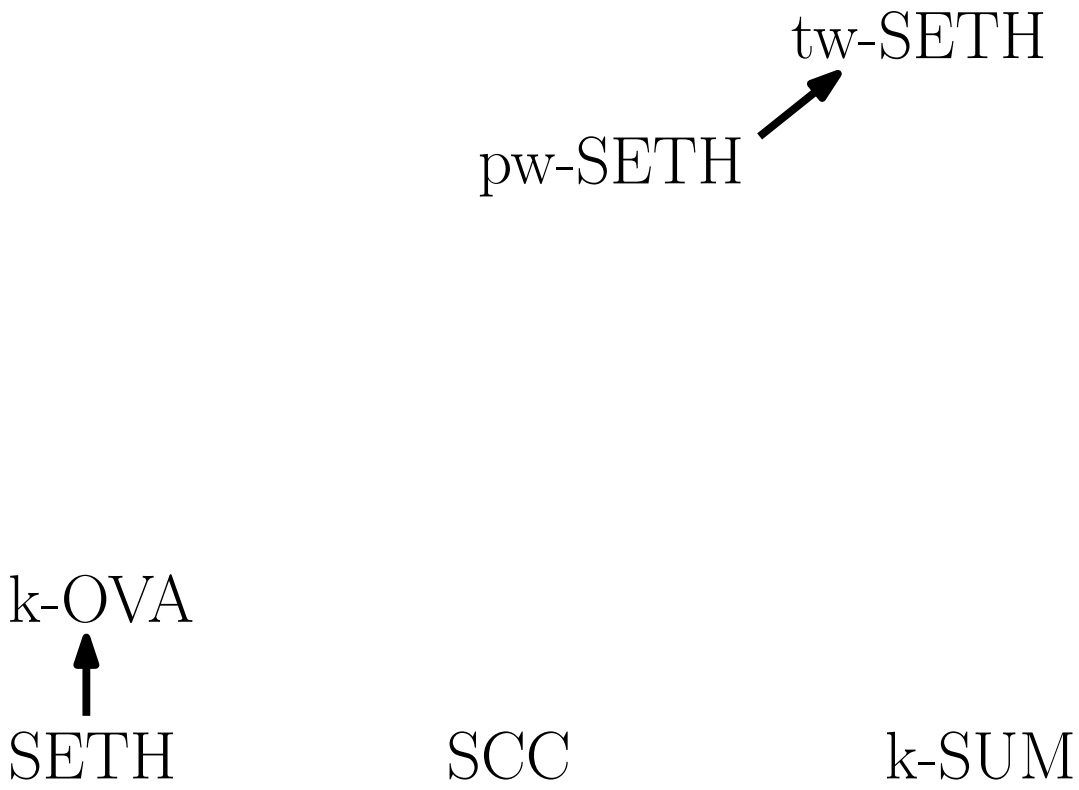
Image credit: nano banana (including typo!)

Motivation and Context

- Many different **flavors** of complexity hypotheses
  - SETH: k-SAT needs brute-force
  - Set Cover Conjecture (SCC): Set Cover needs brute-force DP
  - k-SUM/3-SUM: Meet in the middle is **optimal**
- Some more **credible** than others(?)
  - But typically few implications/relations known
- Some more **relevant** than others
  - SETH (and SCC) **main** hypotheses for **Parameterized Complexity**

Hypotheses

Hypotheses



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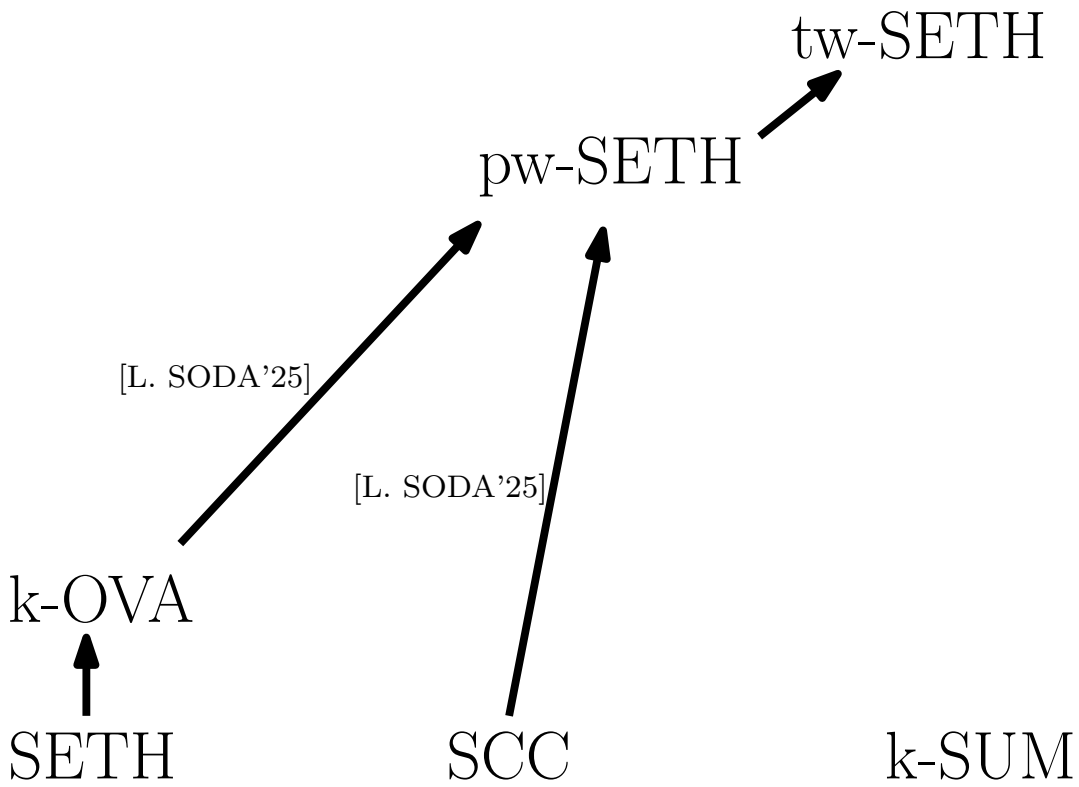

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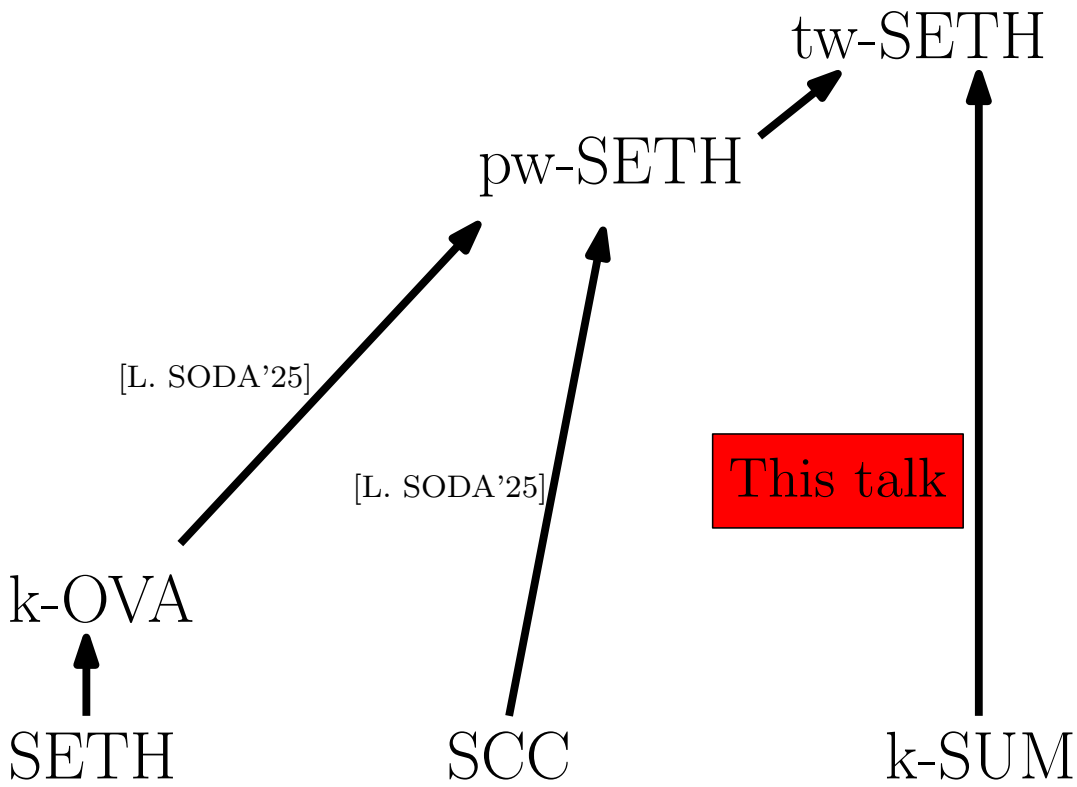

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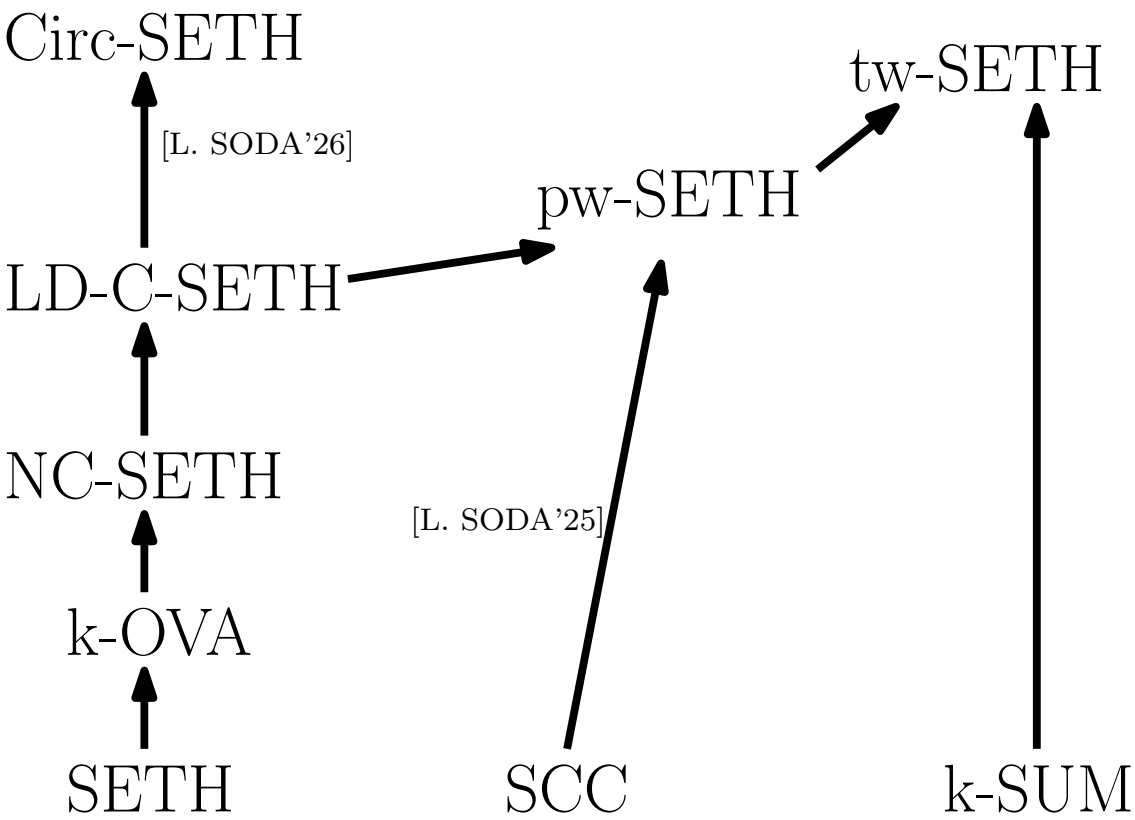
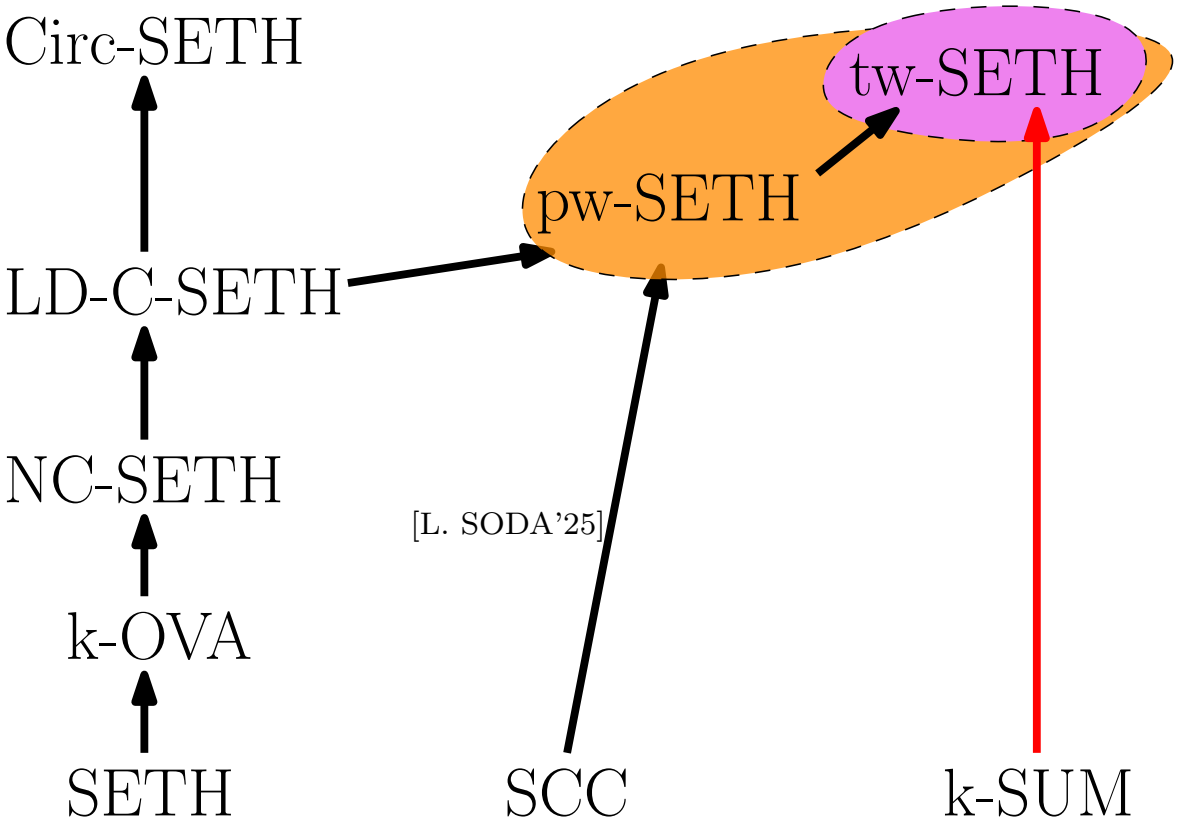

Circ-SETH
[L. SODA'26]
LD-C-SETH
NC-SETH
k-OVA
SETH
pw-SETH
[L. SODA'25]
SCC
tw-SETH
k-SUM

Hypotheses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Proof Sketch

Setup

Definition

In k-SUM, we are given k arrays $A_1, \ldots, A_k$ of n integers each and are asked if it is possible to select $i_1, \ldots, i_k \in [n]$ such that $\sum_j A_j[i_j] = 0$.

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$x_{1,7}$	$x_{2,7}$	$x_{3,7}$	$x_{4,7}$	$x_{5,7}$	$x_{6,7}$
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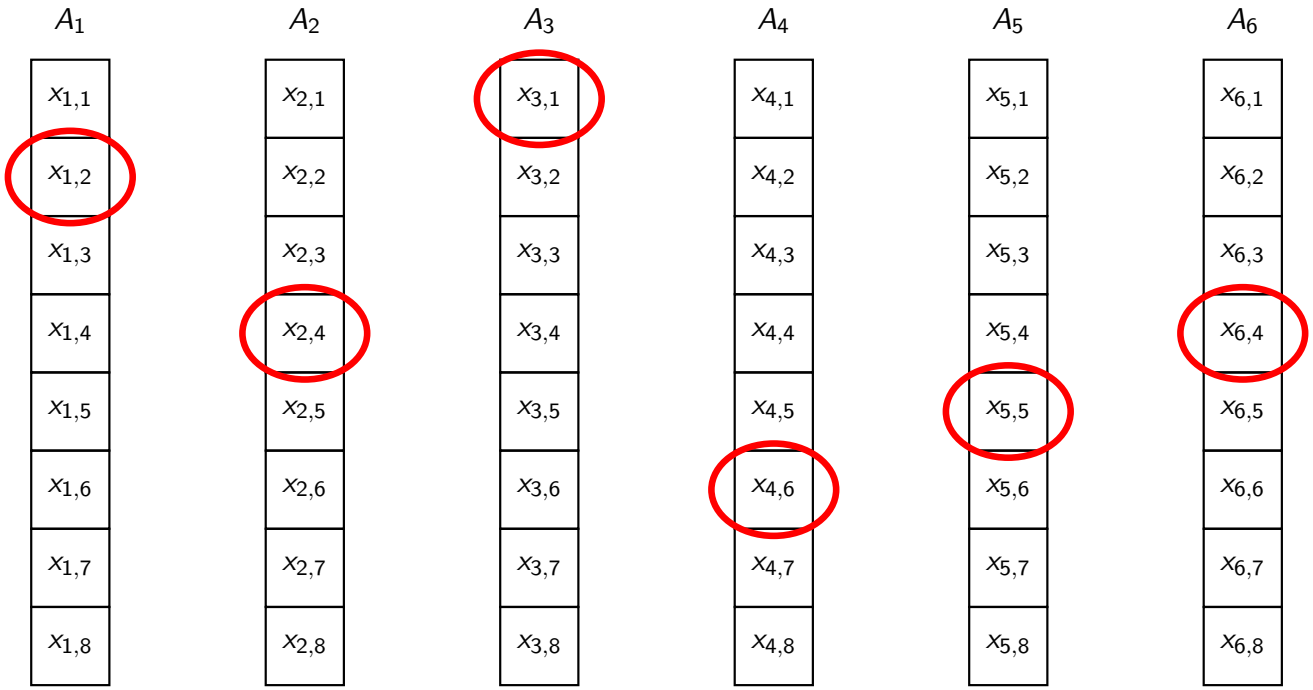

Sum$= 0$?

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High-level strategy:

- Reduce the given instance to a SAT formula of small treewidth
- Show that fast algorithm for such formulas $\Rightarrow$ fast algorithm for k-SUM.

Past success is not a guarantee

Definition

In k-OV, we are given k arrays $A_1, \ldots, A_k$ of n d-bit vectors each and are asked if it is possible to select $i_1, \ldots, i_k \in [n]$ such that $\prod_j A_j[i_j] = 0$.

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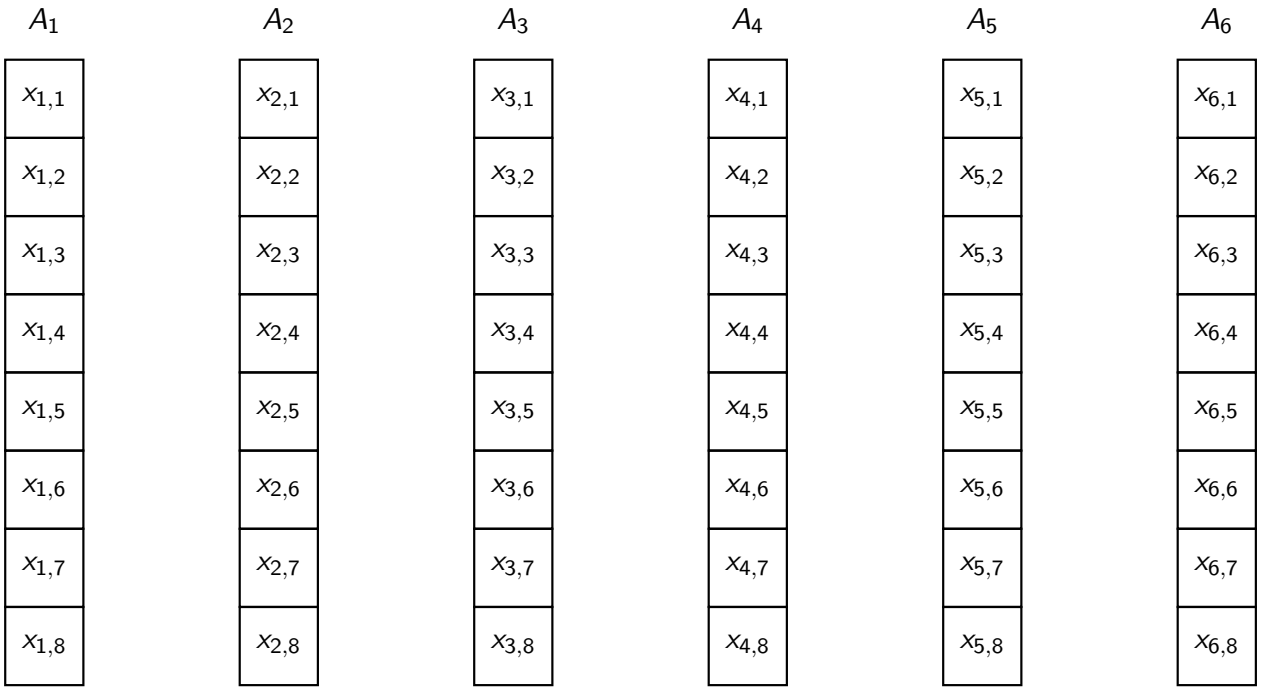

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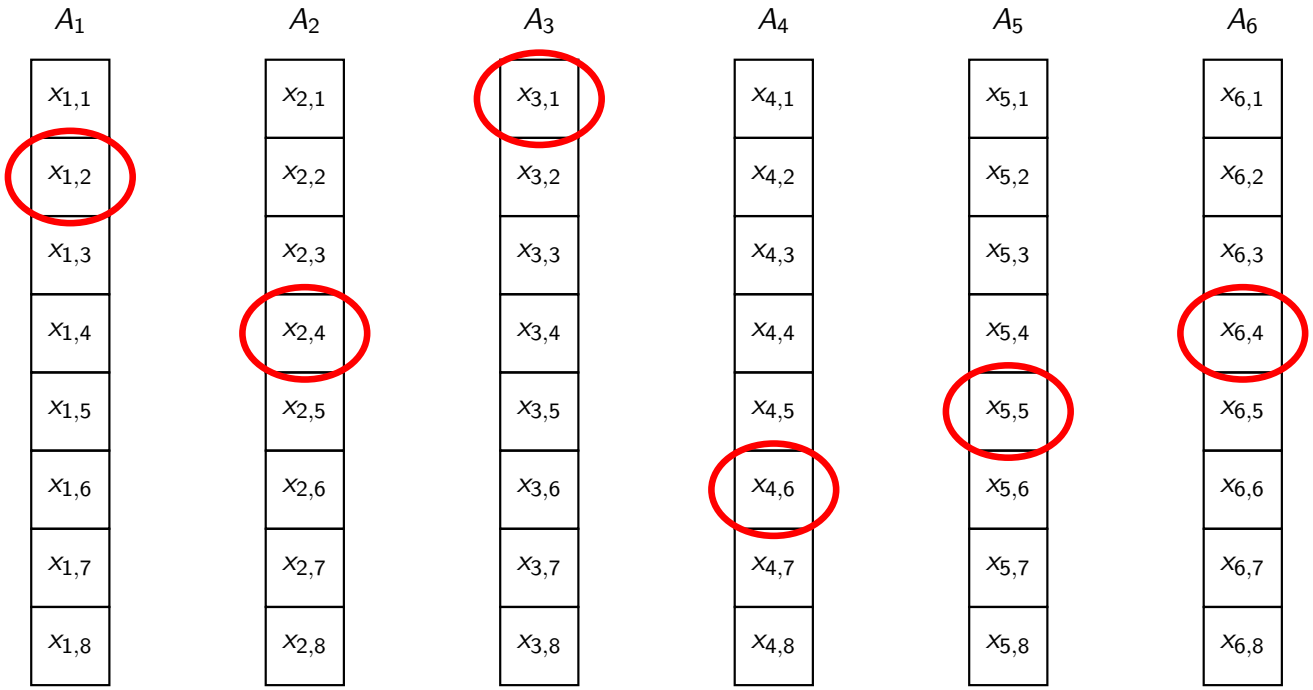

Product$= 0$?

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Looks similar?

Image credit: nano banana

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- Construct a SAT formula with $k \log n$ **main** variables.
- Each group of $\log n$ variables encodes a different i_j.
- Add d groups of vars/clauses to check that each bit of the result is 0.

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- $(2-\varepsilon)^{\mathrm{pw}} n^{O(1)}$ algorithm for $\mathrm{SAT} \Rightarrow n^{(1-\varepsilon)k}$ algorithm for k-OV

[L. SODA'25]

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  [L. SODA'25]

- Great! Why don't we do the same for k-SUM?

Past success is not a guarantee

Definition

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- Construct …s.
- Each grou…
- Add d gro… of the result is 0.
- Pathwidth …
- $(2-\varepsilon)^{\mathrm{pw}} n$… …m for k-OV [L. SODA'25]
- Great! Wh…

Meet in the middle

Meet in the middle

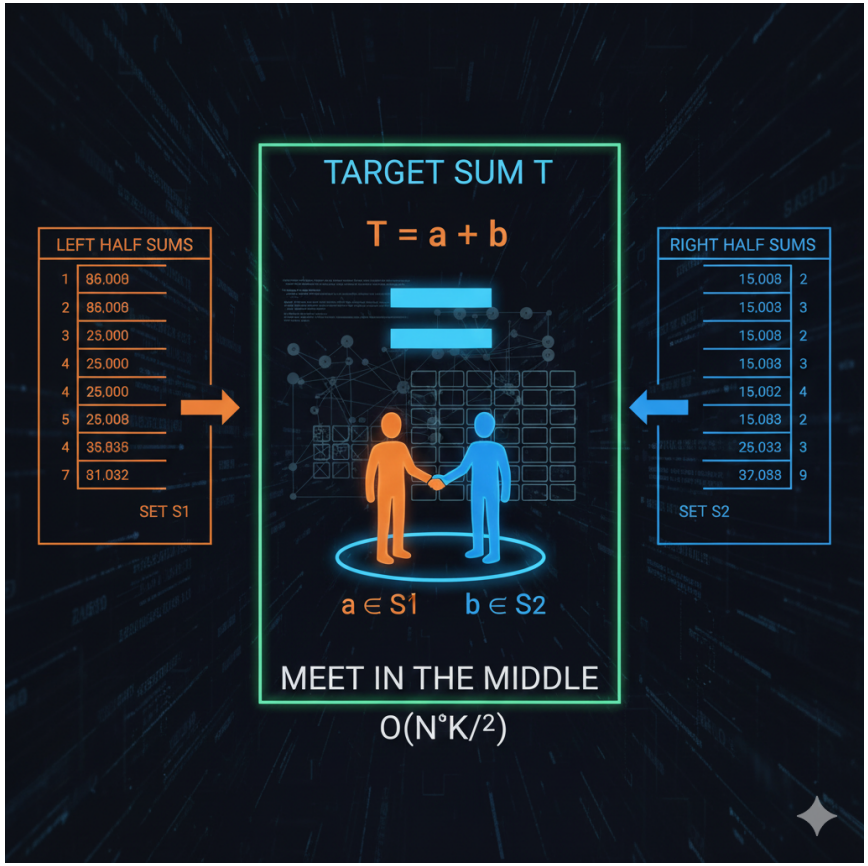

Image credit: nano banana (including **nonsensical** math)

Meet in the middle

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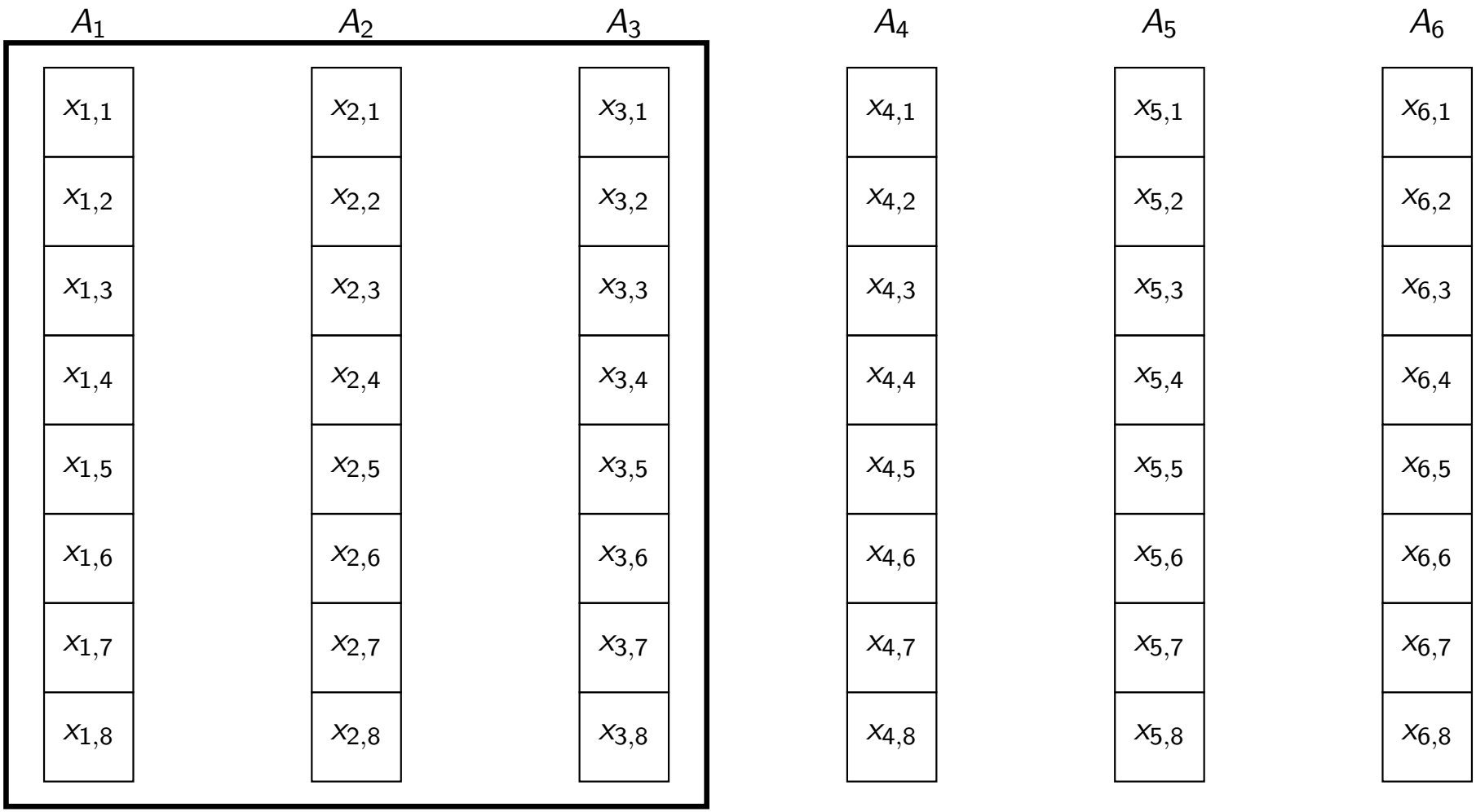

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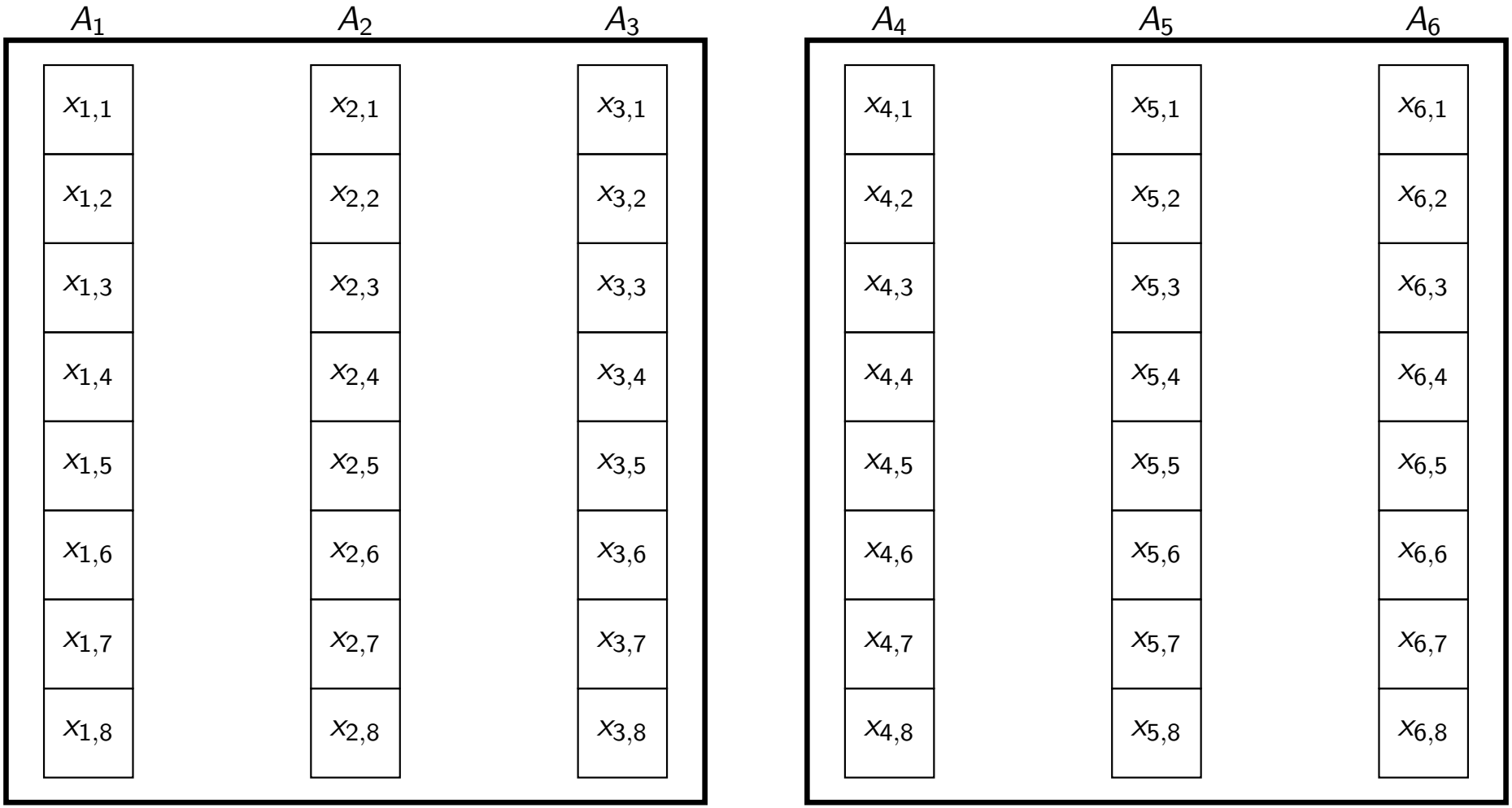

Meet in the middle

Algorithm:

- Construct two **large** arrays (size $n^{k/2}$):
  - L has all sums constructible from $A_1, \ldots, A_{k/2}$
  - R has all sums constructible from $A_{k/2+1}, \ldots, A_k$
- Check if exists $x \in L, y \in R$ such that $x + y = 0$
  - Equivalently: check if $L \cap R \neq \emptyset$
- Can be done in time $\widetilde{O}(n^{k/2})$.

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- Can be done in time $\widetilde{O}(n^{k/2})$.

- **Consequence:** we need a reduction to SAT which produces a formula of width $\frac{k}{2} \log n$ (**not** $k \log n$)
- Running standard DP on this formula should be **as fast as** meet-in-the middle
  - Otherwise, a small improvement on the SAT algorithm will not beat meet-in-the-middle...

Machines and intuition

Machine Characterizations of pw-SETH and tw-SETH

- pw-SETH: Non-deterministic TM with space $\mathrm{pw} + O(\log n)$ [Iwata Yoshida ESA'15]
- tw-SETH: Alternating TM with space $\mathrm{pw} + O(\log n)$ and $O(\log n)$ alternations

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Is k-SUM in NL?

- Yes, but needs space $k \log n$...

Machines and intuition

nd tw-SETH

n space $\mathrm{pw} + O(\log n)$ [Iwata

$\mathrm{pw} + O(\log n)$ and $O(\log n)$

Low Space NDTM for k-SUM

How little memory can an NDTM use to solve k-SUM?

- Approach 1: guess solution indices, then verify
  - Need $k \log n$ bits to store solution...

Low Space NDTM for k-SUM

How little memory can an NDTM use to solve k-SUM?

- Approach 1: guess solution indices, then verify
  - Need $k \log n$ bits to store solution...
- Approach 2: keep track of current sum, for each element non-deterministically check whether to take it.
  - Space needed: $\log W + O(\log n)$, where W is max absolute value
  - Is this better??

Hash me if you can!

Strategy:

- Find a hash function $h : \mathbb{N} \to \{-W, \ldots, W\}$ such that
  - h is (almost-)linear: $h(x + y) \approx h(x) + h(y)$
  - h is "random-looking"
  - W is **small** (ideally $n^{k/2}$)

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  - W is **small** (ideally $n^{k/2}$)
- Almost-linearity guarantees no false negatives
- Good hash function $\Rightarrow$ no false positives
- NDTM needs $\log W + O(\log n) = \frac{k}{2} \log n + O(\log n)$ space
- **Therefore:** k-SUM$\Rightarrow$ pw-SETH

Hash me if you can!

such that

$O(\log n)$ space

Doesn't work!

How much can you hash?

What is the smallest viable value of W?

- Recall $L \cup R$ has size $O(n^{k/2})$
- We need to hash preserving the answer to $L \cap R \neq \emptyset$?
- There are $|L| \times |R|$ possible **bad** pairs

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- There are $|L| \times |R|$ possible **bad** pairs
- If we set $W > n^k$ (and h is "random") whp no pair collides
- However, if $W < n^k$, we will get false positives whp!!

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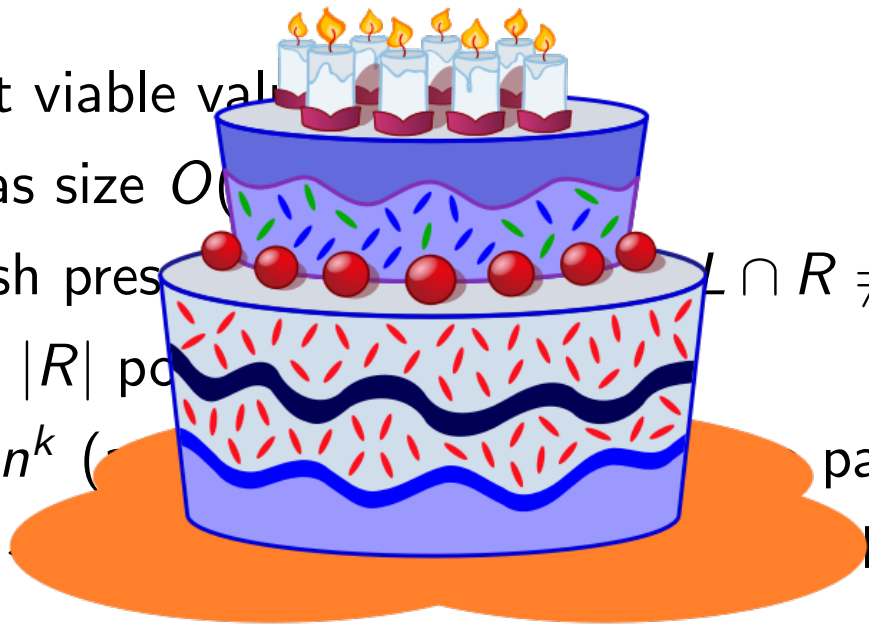


Birthday Paradox!

Perfect Hashing with Arthur and Merlin

Perfect Hashing with Arthur and Merlin

Image credit: ChatGPT

Perfect Hashing with Arthur and Merlin

Perfect Hashing:

- Can I hash N items into a range of size $O(N)$ without collisions?
  - In one shot, **No**, because of birthday paradox problem we saw
  - In two steps, **Yes**!
    - ⋆ (Fredman, Komlós, Szemerédi J.ACM'84)
  - Key intuition: run through a first hash function, each bucket will have few collisions, run through a second hash function to sort them out.

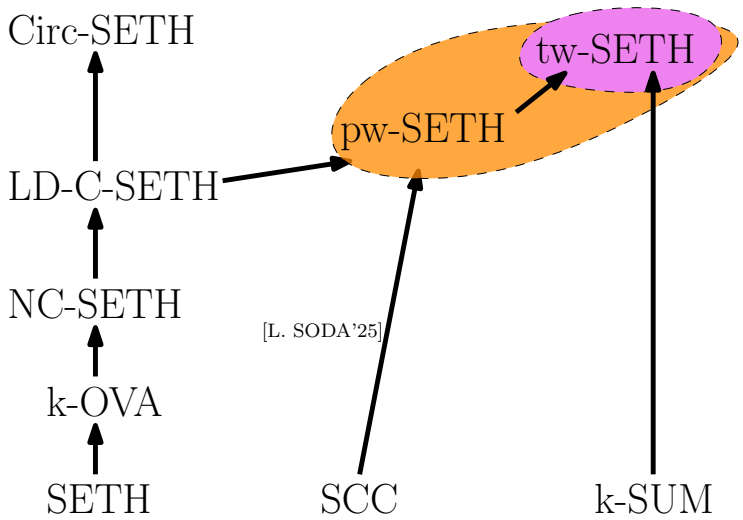
Perfect Hashing with Arthur and Merlin

Protocol:

- Apply an almost-linear hash $h : \mathbb{N} \to \{-W, \ldots, W\}$, with $W = n^{k/2}$
  - Will almost surely have **false positives**
- Challenge the prover to give $x \in L, y \in R$ with $h(x) = h(y)$
  - Prover **commits** to $h(x) = h(y)$, which we **store in memory**
- Pick a second hash function $h' : \mathbb{N} \to \{-w, \ldots, w\}$, where w is large enough to perfectly hash **a single bucket**
- Challenge prover to give $x \in L, y \in R$ with
  - $h(x) = h(y)$ as before
  - $h'(x) = h'(y)$
- **Key Intution**: Treewidth $\Leftrightarrow$ Alternating TM $\Leftrightarrow$ Interactive Proof

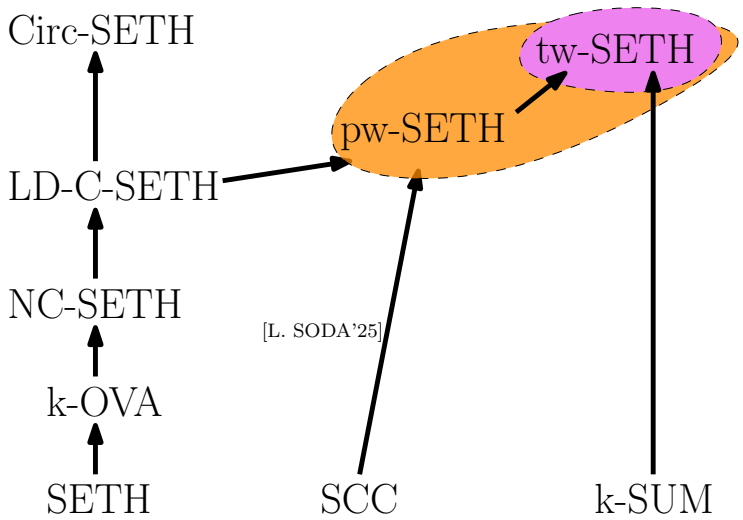
Open Questions

Open Questions



- k-SUM⇒pw-SETH?
- Other connections?

Open Questions



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Thank you!