The smallest 5-chromatic tournament

Thomas Bellitto

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Sorbonne Université, LIP6, Paris, France

Joint work with
Nicolas Bousquet, Université Lyon 1, LIRIS, France
Adam Kabela, University of West Bohemia, Pilsen, Czech Republic
Théo Pierron, Université Lyon 1, LIRIS, France

1. Introduction
2. Our results
   - Tournaments on 12 vertices
   - Tournaments on 17 vertices
   - Tournaments on 18 vertices
   - Tournaments on 19 vertices

Directed coloring

Chromatic number

The chromatic number of a graph G is the smallest number of colors required to assign a color to each vertex of the graph so that no color class contains an edge.

Directed coloring

Directed chromatic number (Neumann-Lara, 1982)

The directed chromatic number of a digraph D is the smallest number of colors required to assign a color to each vertex of the graph so that no color class contains a closed walk.

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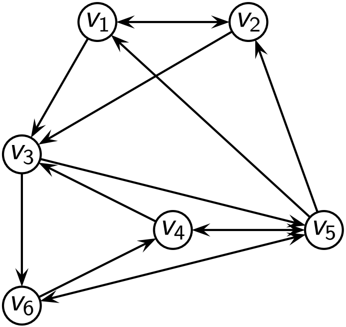

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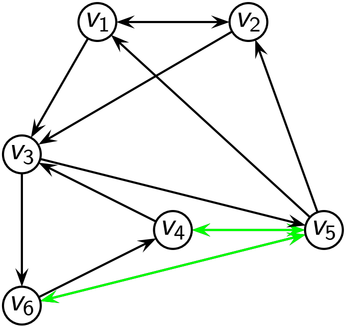

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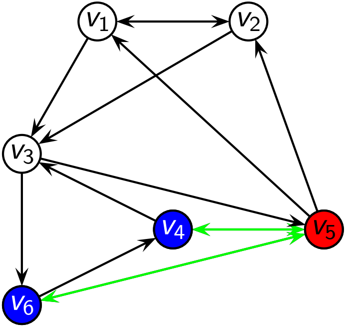

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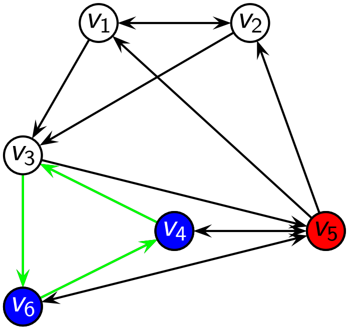

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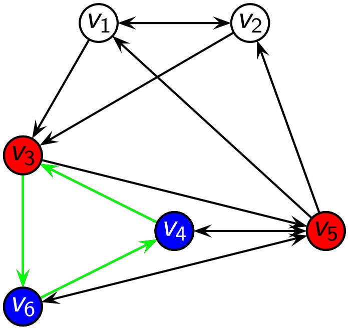

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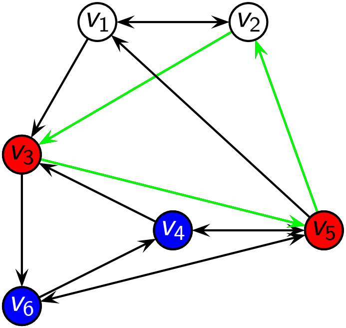

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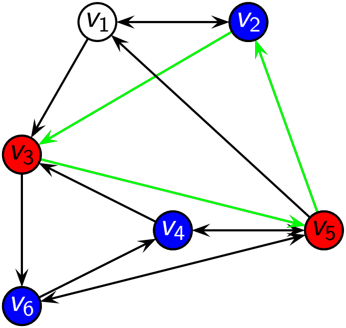

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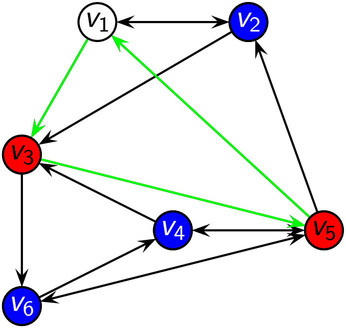

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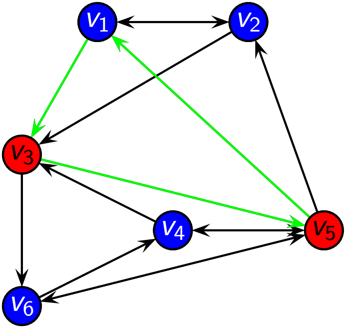

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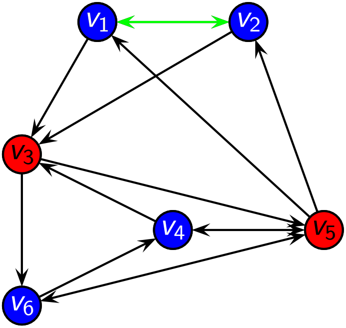

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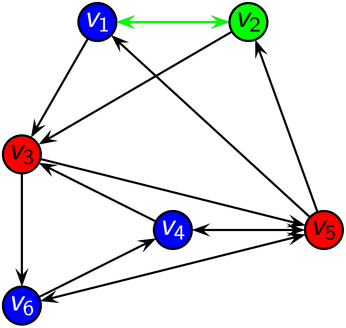

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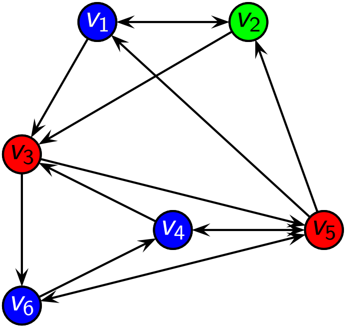

A few definitions

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An oriented graph is a directed graph such that for every pair of vertices u, v, there is at most one arc between u and v.

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Transitive tournament = Acyclic tournament
Proper coloring of a tournament = partition into transitive subtournaments

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Neumann-Lara

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- Can be restricted to tournaments.
- Oriented graphs are important to understand directed coloring, many problem study them. (cf next talk by Clément Rambaud about the smallest number of edges in 3-critical oriented graph)
- Numerous works about the size of the smallest undirected triangle-free graph of chromatic number k (open for $k \geqslant 6$).

Paley tournaments

If $n = 4k + 3$ is prime, the Paley tournament on n vertices P_n is the tournament such that there is an arc from i to j iff $j - i$ is a square $\mod n$.

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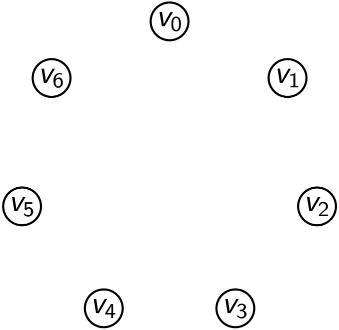

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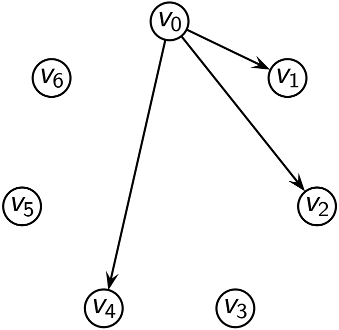

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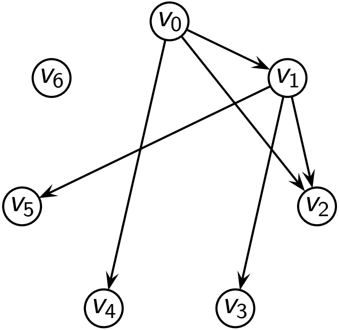

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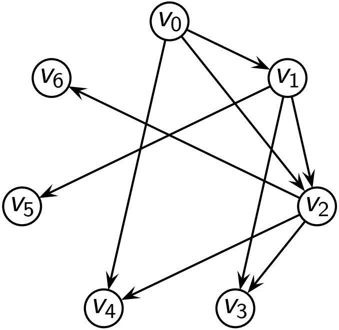

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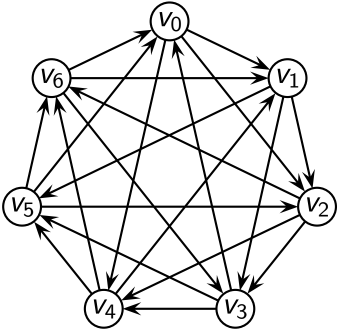

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- Neumann-Lara, 1994: $n_4 = 11$.
  The only such tournament is P_{11}.
- Erdős, 1979: maximum dichromatic number of a tournament on n vertices is $\Theta(\frac{n}{\log n})$.
- Neumann-Lara, proof not given: $17 \leqslant n_5 \leqslant 19$.
  But P_{19} has chromatic number 4.
  Conjecture (Neumann-Lara, 1994): $n_5 = 17$.

Can we bruteface it?

Up to isomorphisms, there are

- 244912778438520759443245824 (27 digits) tournaments on 17 vertices;
- 1783398846284777975419600287232 (31 digits) tournaments on 18 vertices;
- 24605641171260376770598003978281472 (35 digits) tournaments on 19 vertices.

Enumerating them up to isomorphisms is difficult.
We have to solve an NP-complete problem on each of them.

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1 Introduction

2 Our results

- Tournaments on 12 vertices
- Tournaments on 17 vertices
- Tournaments on 18 vertices
- Tournaments on 19 vertices

Structure

Theorem (Sanchez-Flores, 1998)

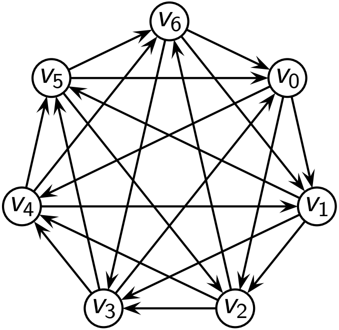
There is a unique tournament on 12 vertices that does not contain a TT_5 and it is 3-chromatic.

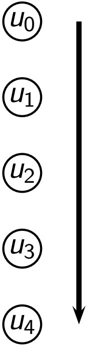
Consequence

In every 4-chromatic tournament on 12 vertices, there is a TT_5 whose removal yields one of the four 3-chromatic tournament on 7 vertices.

Method

P_7 TT_5





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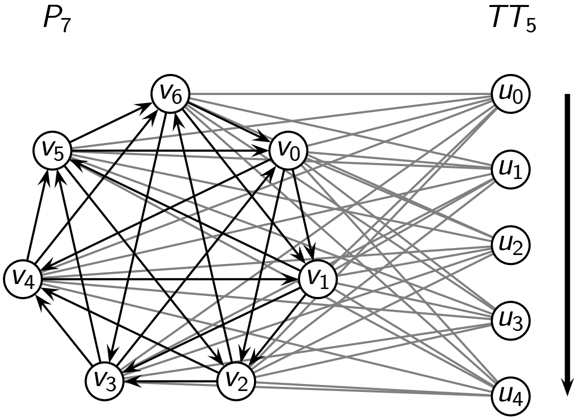

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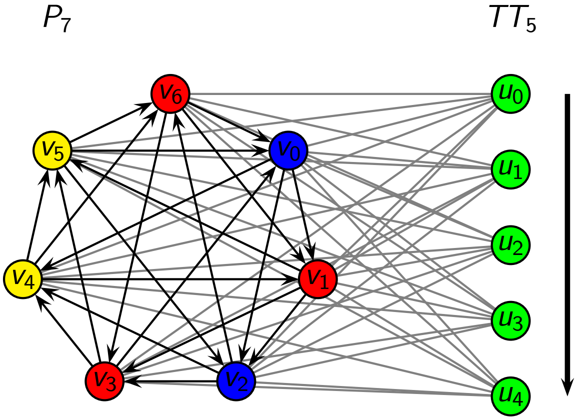

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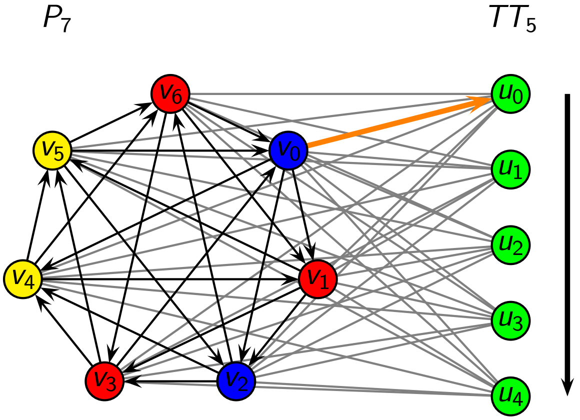

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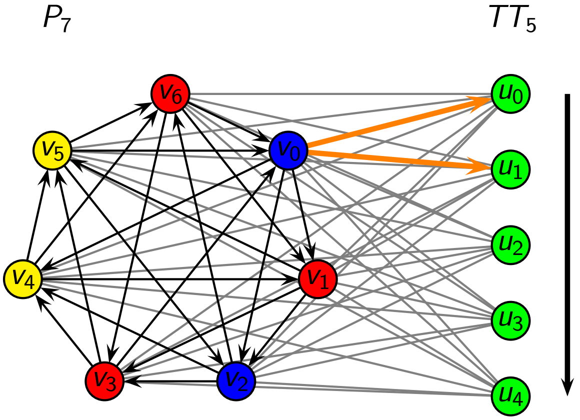

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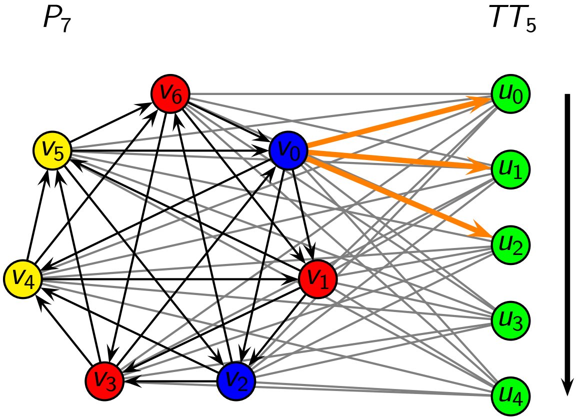

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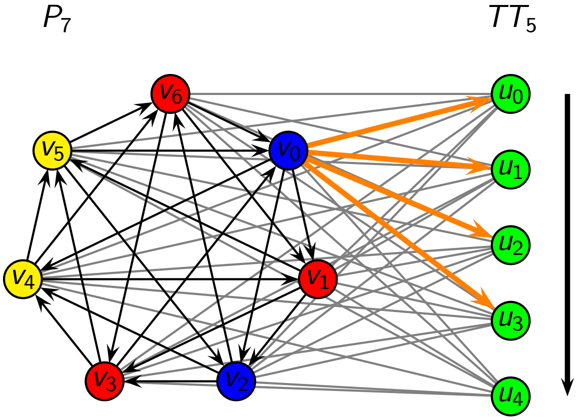

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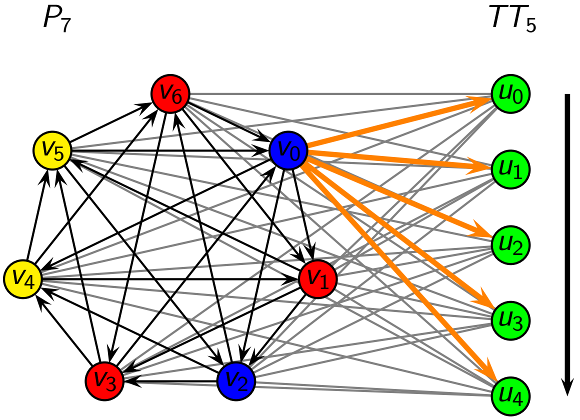

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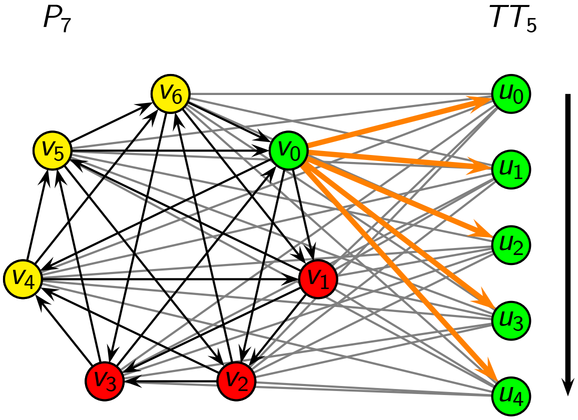

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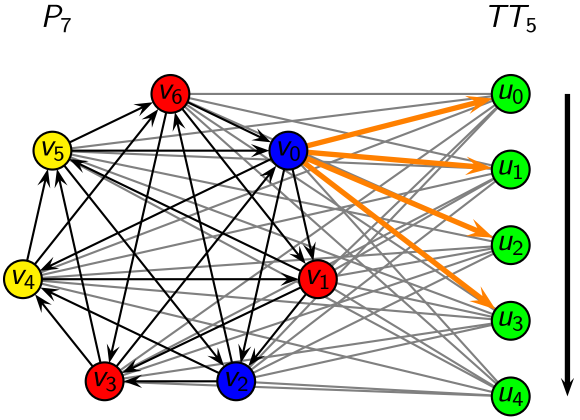

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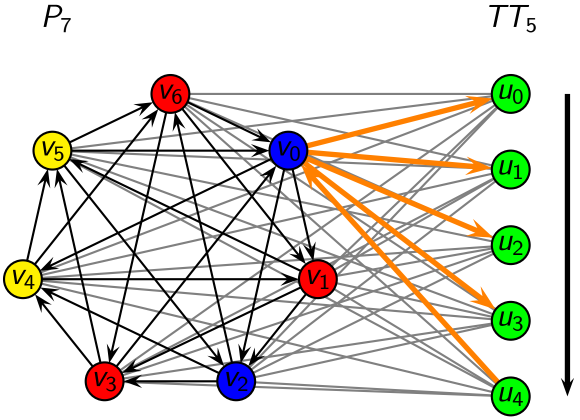

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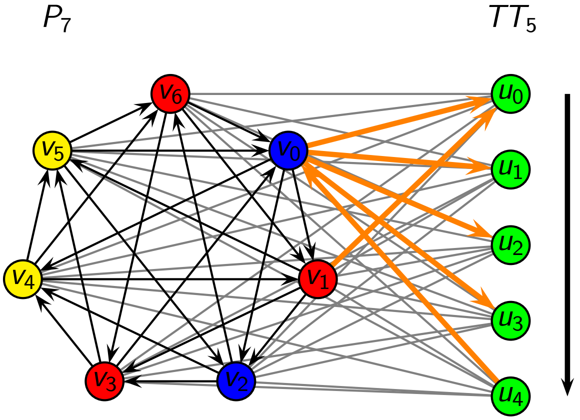

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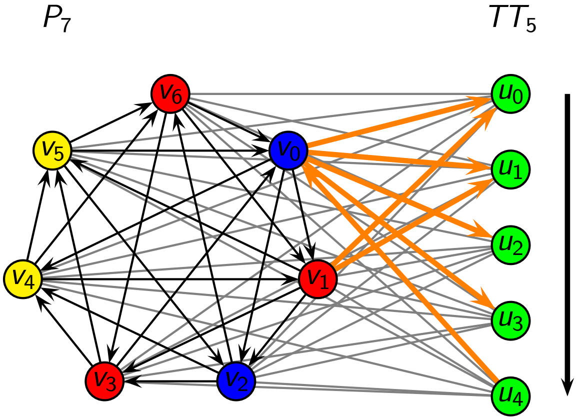


Results

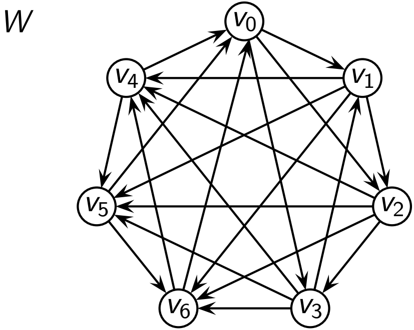
Theorem (Bellitto, Bousquet, Kabela, Pierron)

Every 4-chromatic tournament on 12 vertices :

- contains P_{11};
- is a junction of TT_5 and W_1 (the 3-chromatic 7-vertex tournament contained by P_{11}).

There are 3-chromatic tournaments on 8 vertices that do not contain any 3-chromatic tournaments on 7 vertices.

W_1



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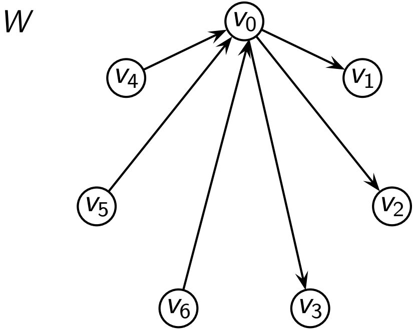

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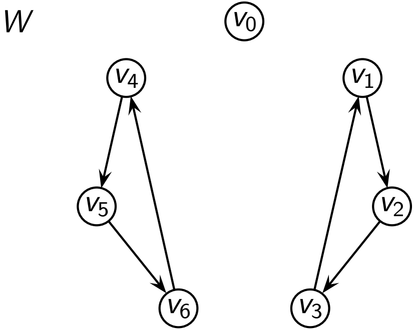

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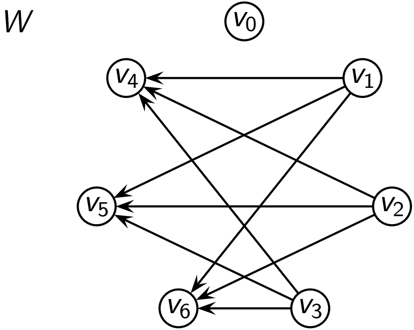

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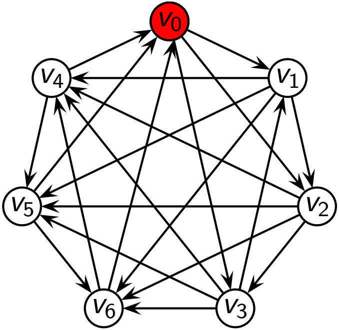

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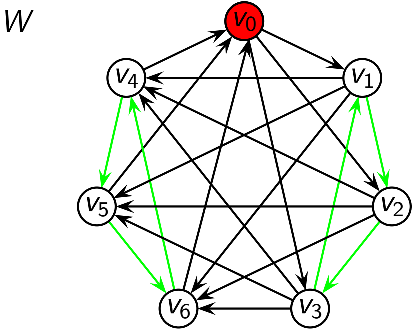

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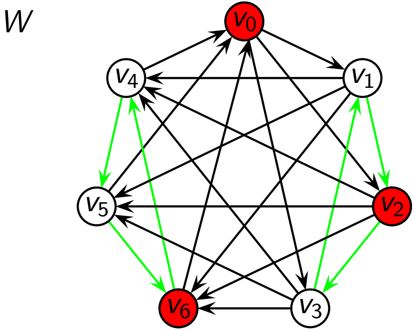

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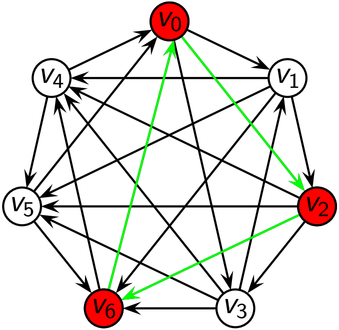

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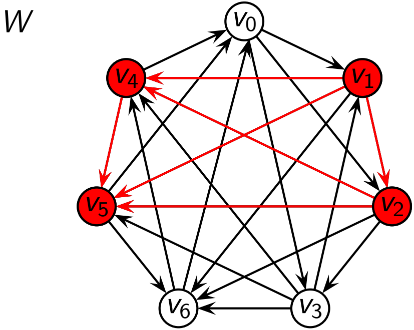

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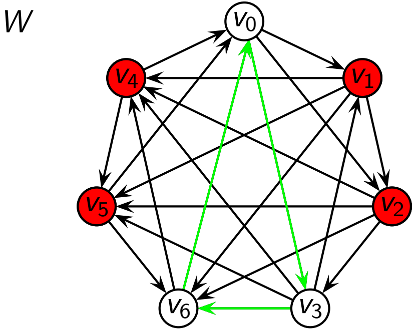

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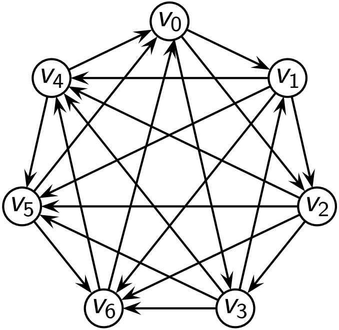

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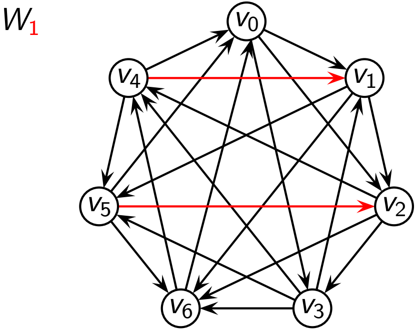

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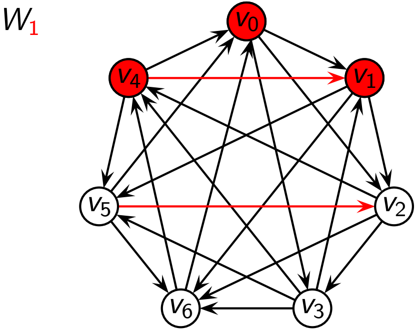

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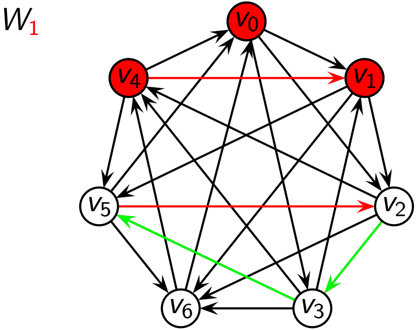

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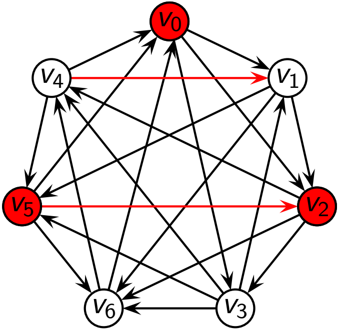

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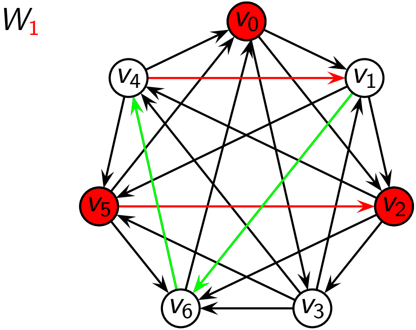

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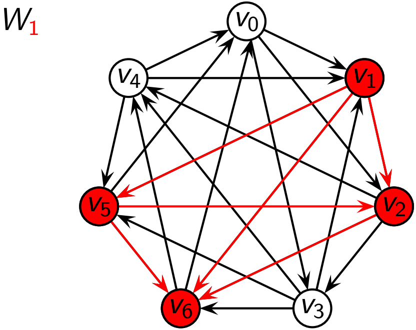

Consequences

Critical graph

A graph is *k-critical* iff its chromatic number is k but drops to $k-1$ if you remove any vertex or arc.
It is *k-vertex-critical* iff its chromatic number drops to $k-1$ if you remove a vertex.

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It is *k-vertex-critical* iff its chromatic number drops to $k-1$ if you remove a vertex.

Theorem (Aboulker, Bellitto, Havet, Rambaud)

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Consequences

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Theorem (Aboulker, Bellitto, Havet, Rambaud)

There is no 4-critical oriented graph on 12 vertices.
$p_k \neq n_k$

Outline of the proof

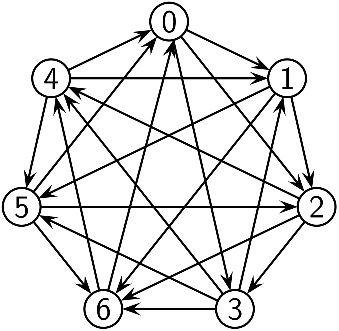
Structure of the graph

If T is a 5-chromatic tournament on 17 vertices, then we can partition its vertices into A_1, A_2 and B such that

- A_1 and A_2 induce two copies of TT_5
- B induces a copy of W_1

But...

One can always partition B into B_1 and B_2 such that the tournaments induced by $A_1 \cup B_1$ and $A_2 \cup B_2$ are 2-colorable.



- $\chi(A_i \cup \{0,1,4\}) = 2$.
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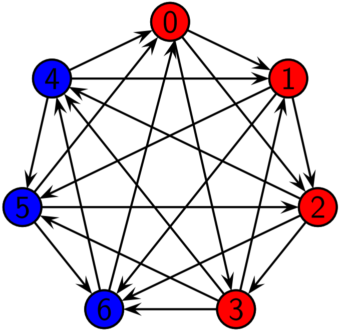

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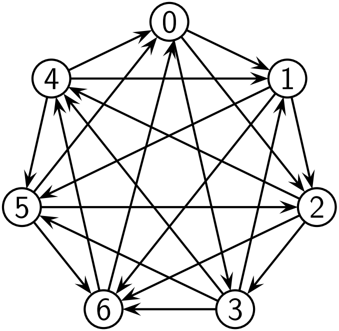

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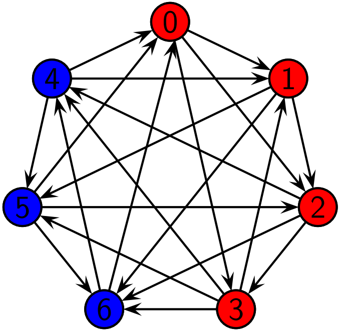

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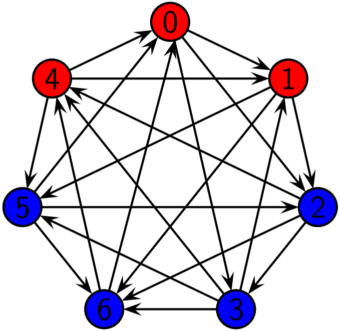

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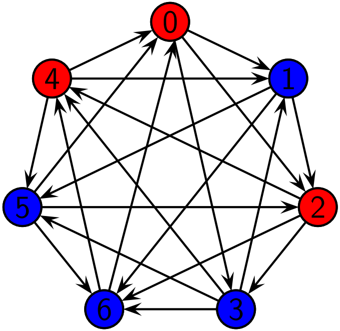

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Let T be 5-chromatic on 18 vertices.

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Let T be 5-chromatic on 18 vertices. If T has 3 disjoint TT_5

- We build all the 3-chromatic 8-vertex tournaments we can by joining C_3 and TT_5.
- We build all the 4-chromatic 13-vertex tournaments we can by joining C_3 and 2 TT_5.
- We cannot build any 5-chromatic 18-vertex tournaments by joining C_3 and 3 TT_5.

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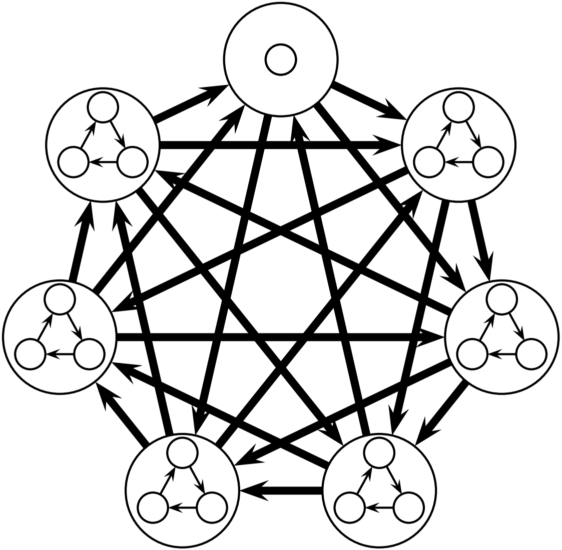
If T has 2 disjoint TT_5

- Same idea as previous section but B induces one of the 94 3-chromatic 8-vertex TT_5-free tournaments.

A 5-chromatic tournament on 19 vertices



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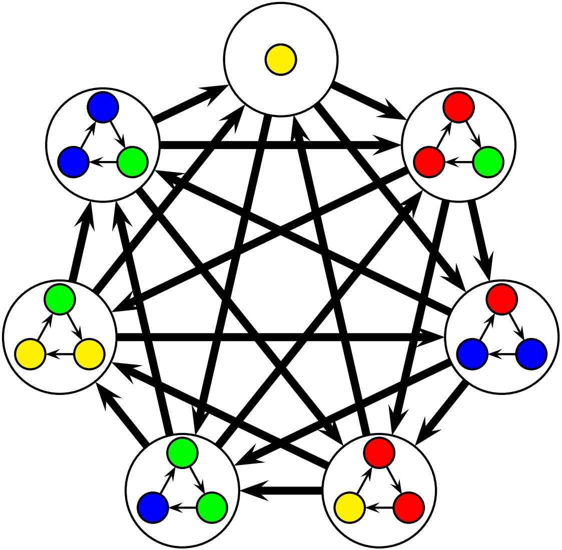

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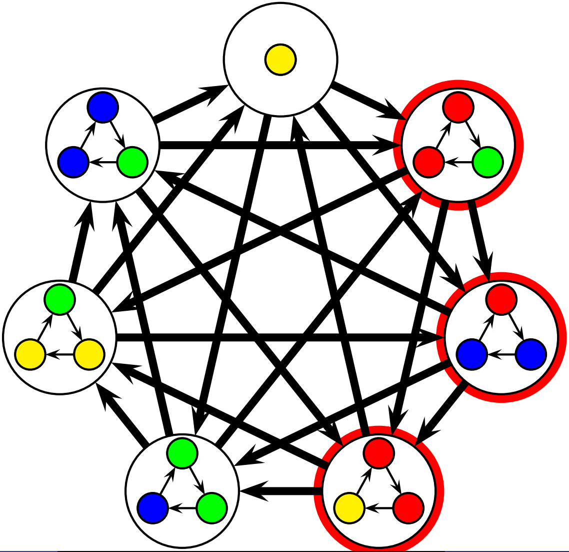

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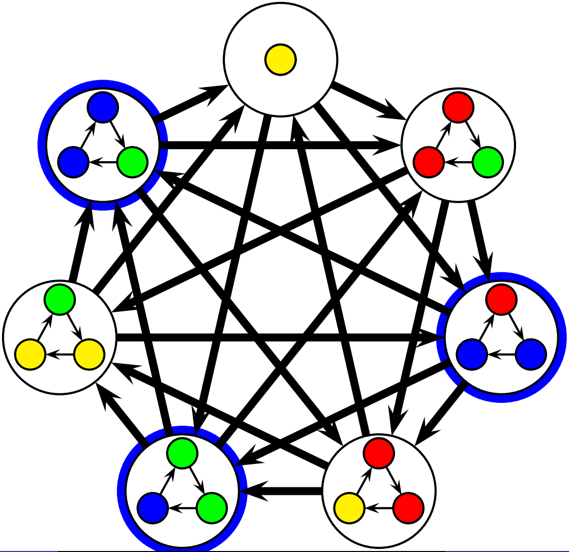

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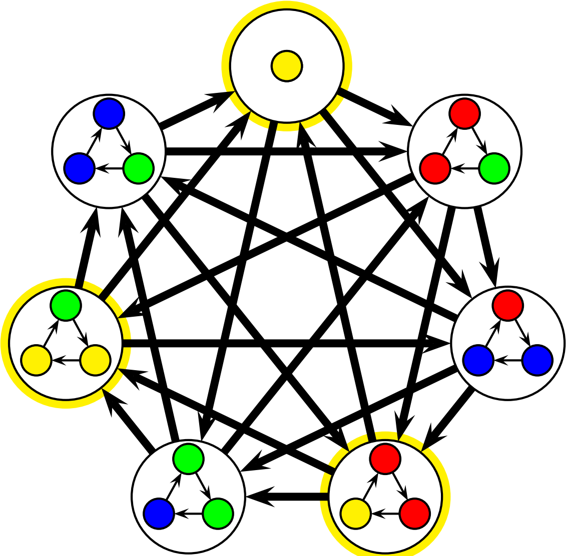

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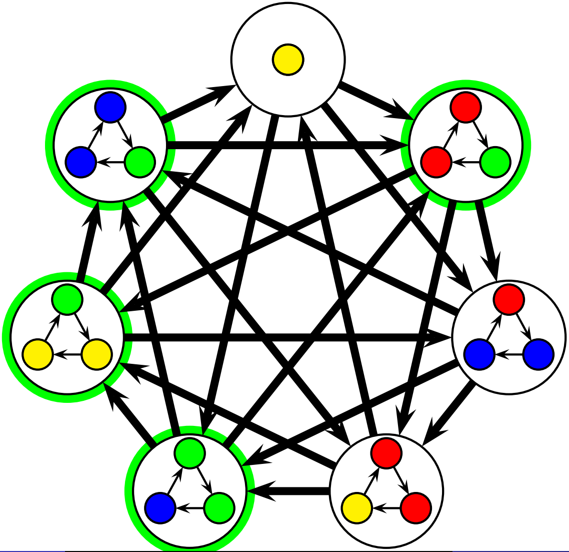

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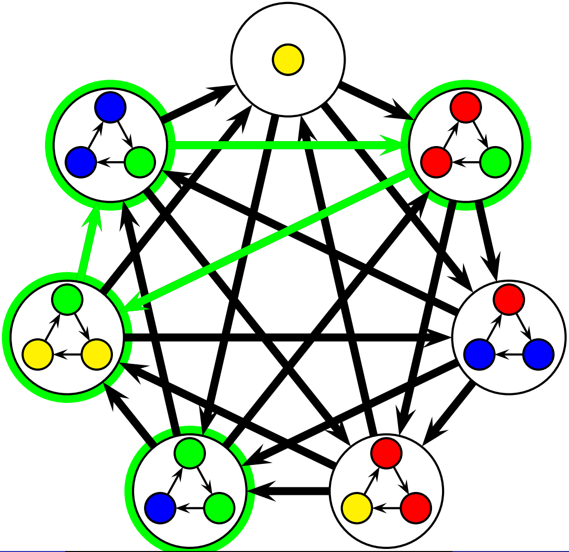

Perspectives

- Combinatorial proof that there are no vertex-critical tournament on 12 vertices.
- Counting/enumerating the 5-chromatic 19-vertex tournaments?
- Computing n_6 seems currently out of reach.

Thank you!