

Weighted independence ratio of geometric distance graphs

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Joint work with

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- 1 Context
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 - The Euclidean plane
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- 2 Polytope norms in the plane
 - The problem
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 - Relation to fractional colouring
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Definitions

- Normed space $E = (\mathbb{R}^n, \|\cdot\|)$.
- A set $A \in \mathbb{R}^n$ avoids distance 1 iff $\forall x, y \in A, \|x - y\| \neq 1$.
- (Upper) density of a measurable set A :

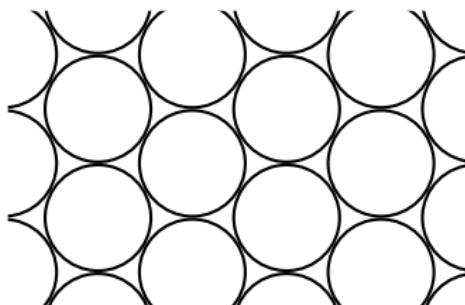
$$\delta = \limsup_{R \rightarrow \infty} \frac{\text{Vol}(A \cap [-R, R]^n)}{\text{Vol}([-R, R]^n)}.$$

- Maximum density of a set avoiding distance 1:

$$m_1(\mathbb{R}^n, \|\cdot\|) = \sup_{A \text{ avoiding } 1} \delta(A).$$

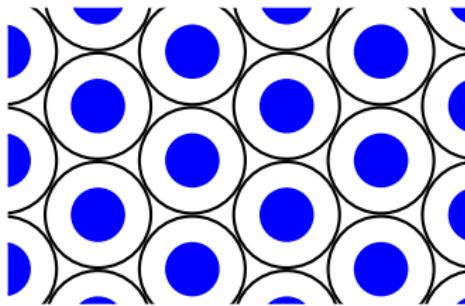
Example

- Let Λ be a set of two pairwise disjoint balls of radius 1.



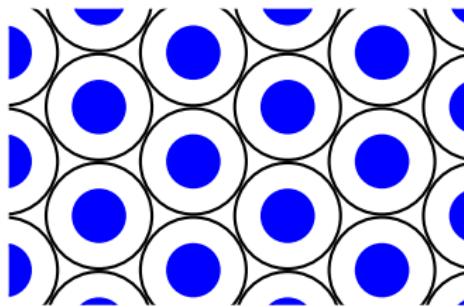
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- We build from Λ a set avoiding distance 1 of density $\frac{\delta(\Lambda)}{2^n}$.



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- If the unit ball associated to a norm $\|\cdot\|$ tiles \mathbb{R}^n ($\|\cdot\|_\infty$ for example) :

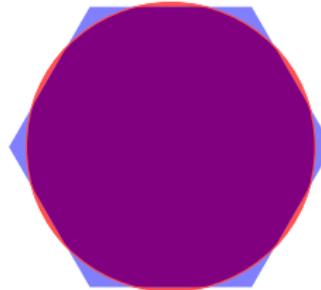
$$m_1(\mathbb{R}^n, \|\cdot\|) \geq \frac{1}{2^n}.$$

Lower bounds

- The previous construction proves that $m_1(\mathbb{R}^2, \|\cdot\|_2) \geq 0.9069/4 \geq 0.2267$.

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- Croft (1967): $m_1(\mathbb{R}^2, \|\cdot\|_2) \geq 0.229$.



Upper bounds

- Best upper bound : $m_1(\mathbb{R}^2, \|\cdot\|_2) \leq 0.258795$ (Keleti, Matolcsi, de Oliveira Filho, Ruzsa, 2015).
- Erdős' conjecture : $m_1(\mathbb{R}^2, \|\cdot\|_2) < 1/4$.
- Generalization (Moser, Larman Rogers):
 $m_1(\mathbb{R}^n, \|\cdot\|_2) < \frac{1}{2^n}$.

Definitions

Chromatic number of a metric space

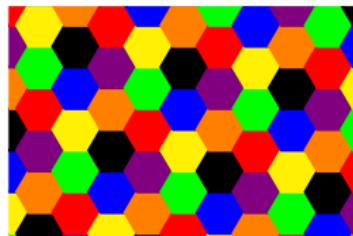
The **chromatic number** χ of a metric space (X, d) is the smallest number of colours required to colour each point of X so that no two points at distance 1 share the same colour.

Unit-distance graph

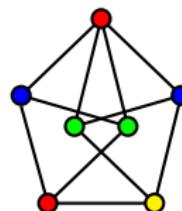
The **unit-distance graph** associated to a metric space (X, d) is the graph G such that $V(G) = X$ and $E(G) = \{\{x, y\} : d(x, y) = 1\}$.

The Euclidean plane

- $\chi(\mathbb{R}^2) \leq 7$:



- $\chi(\mathbb{R}^2) \geq 4$ (Moser's spindle):



- De Grey (April 2018): $\chi(\mathbb{R}^2) \geq 5$.

Measurable chromatic number

We define the **measurable chromatic number** χ_m of a metric space (X, d) by adding the constraint that the colour classes must be measurable set.

$$\chi_m(\mathbb{R}^n, \|\cdot\|) \geq \frac{1}{m_1(\mathbb{R}^n, \|\cdot\|)}$$

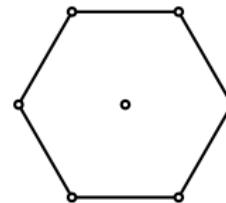
Euclidean plane: $\chi_m(\mathbb{R}^2) \geq 5$. (Falconer, 1981)
Same bound as in the non-measurable case.

Polytope norm

Polytope norm

Let \mathcal{P} be a convex, symmetric polytope centered at 0 and of non-empty interior. The *polytope norm* $\|\cdot\|_{\mathcal{P}}$ associated to \mathcal{P} is by definition

$$\|x\|_{\mathcal{P}} = \inf\{t \in \mathbb{R}^+ : x \in t\mathcal{P}\}.$$

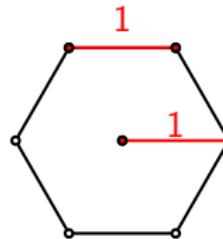


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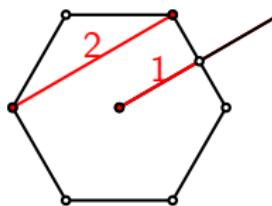


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If the unit ball associated to a norm $\|\cdot\|_{\mathcal{P}}$ is a polytope that tiles \mathbb{R}^n (by translation), $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) \geq \frac{1}{2^n}$.

Conjecture (Bachoc, Robins)

If \mathcal{P} tiles \mathbb{R}^n (by translation), then $m_1(\mathbb{R}^n, \|\cdot\|_{\mathcal{P}}) = \frac{1}{2^n}$.

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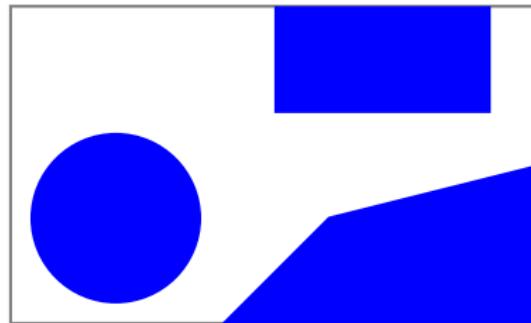
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Theorem (Bachoc, Bellitto, Moustrou, Pêcher, 2017)

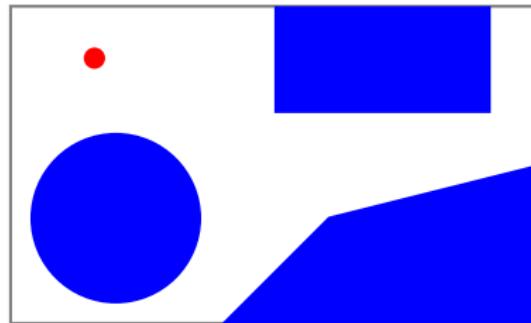
If \mathcal{P} tiles \mathbb{R}^2 (by translation), then $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{P}}) = \frac{1}{4}$.

Method



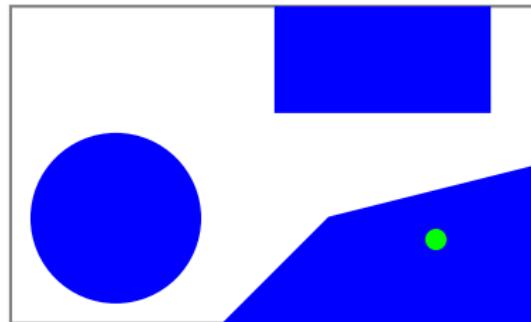
Set S of density δ . X at random in \mathbb{R}^n :

Method



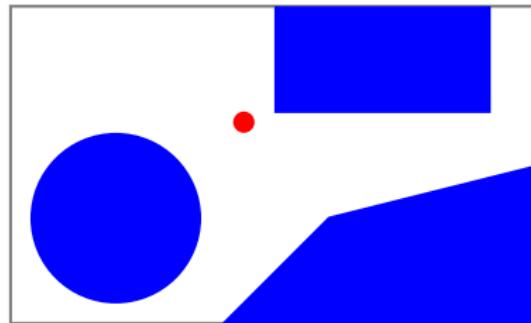
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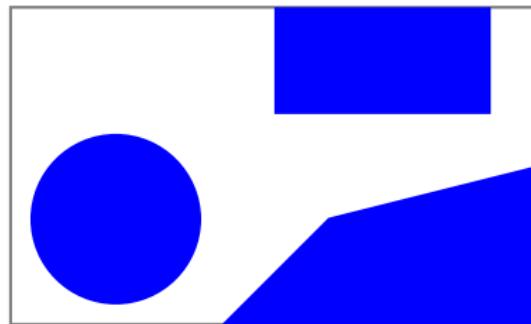
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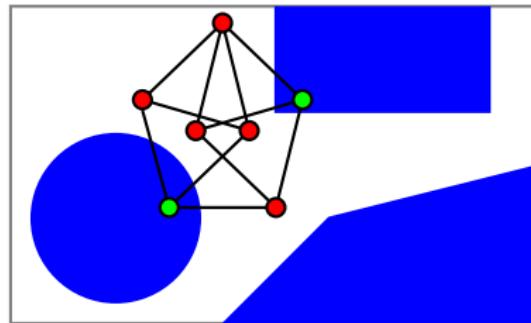
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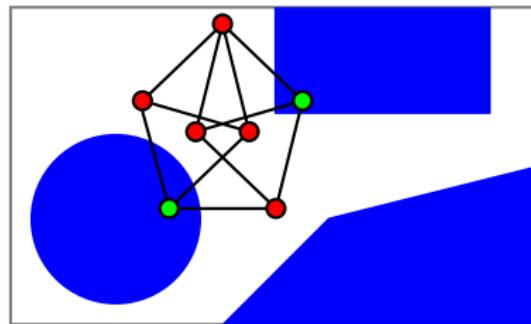
Set S of density δ . X at random in \mathbb{R}^n : $\mathbb{P}(X \in S) = \delta$.

Method



Set S of density δ . X at random in \mathbb{R}^n : $\mathbb{P}(X \in S) = \delta$.
Unit-distance subgraph G at random in \mathbb{R}^n :

Method

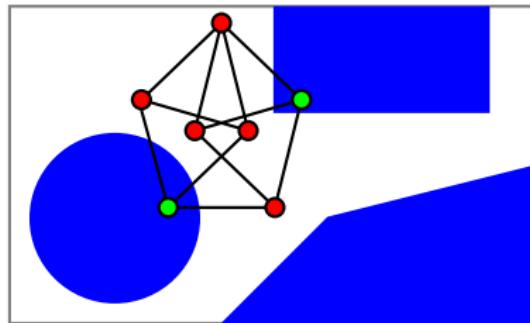


Set S of density δ . X at random in \mathbb{R}^n : $\mathbb{P}(X \in S) = \delta$.

Unit-distance subgraph G at random in \mathbb{R}^n :

$$\mathbb{E}(|V \cap S|) = |V| \times \delta.$$

Method



Set S of density δ . X at random in \mathbb{R}^n : $\mathbb{P}(X \in S) = \delta$.

Unit-distance subgraph G at random in \mathbb{R}^n :

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If S avoids distance 1: $|V \cap S| \leq \alpha(G) \rightarrow \delta \leq \frac{\alpha}{|V|}$.

Discretization lemma

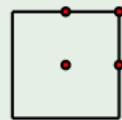
For all unit-distance subgraph G in \mathbb{R}^n :

$$m_1(\mathbb{R}^n) \leq \overline{\alpha(G)} = \frac{\alpha(G)}{|V|}.$$

Discretization lemma

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Determining $m_1(\mathbb{R}^2, \|\cdot\|_\infty)$ 

K_4 is a unit-distance subgraph.

$$m_1(\mathbb{R}^2, \|\cdot\|_\infty) = \frac{1}{4}.$$

Definitions

Weighting of a graph: $w : V \rightarrow \mathbb{R}^+$.

Weight of a vertex set S : $\sum_{v \in S} w(v)$.

Weighted independence number $\alpha_w(G)$ of a weighted graph G : maximum weight of an independent set.

Weighted independence ratio $\overline{\alpha_w(G)} = \frac{\alpha_w(G)}{w(G)}$.

Definitions

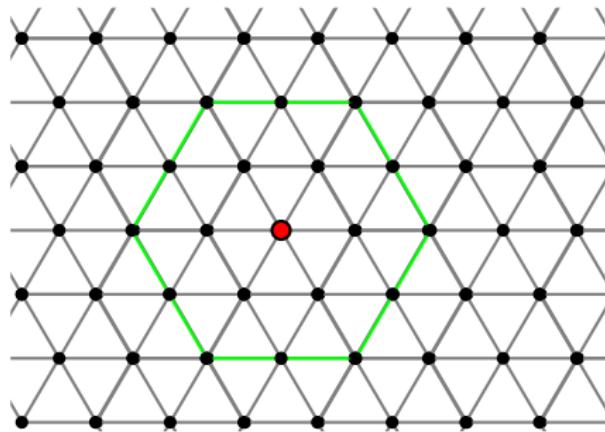
Optimal weighted independence ratio $\alpha^*(G)$ of an unweighted graph G : minimum over all weightings of G of $\alpha(G)$.

Discretization lemma

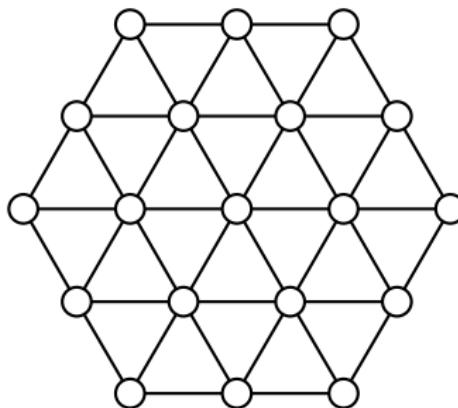
For all unit-distance subgraph G in \mathbb{R}^n :

$$m_1(\mathbb{R}^n) \leq \alpha^*(G).$$

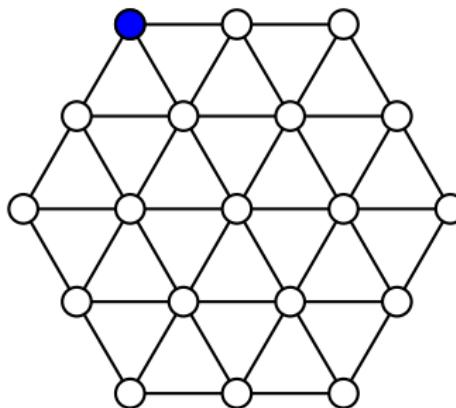
The regular hexagon



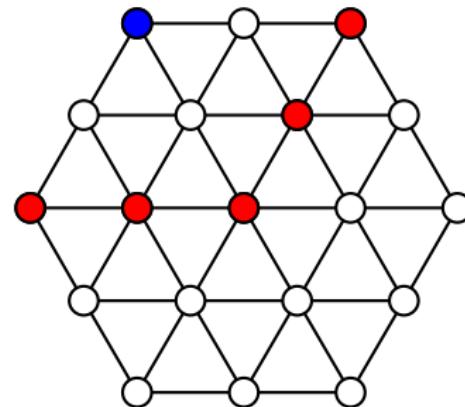
With an unweighted graph



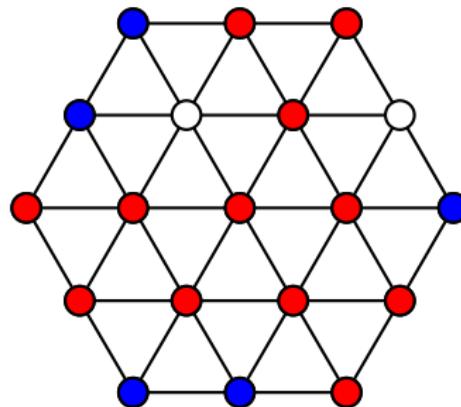
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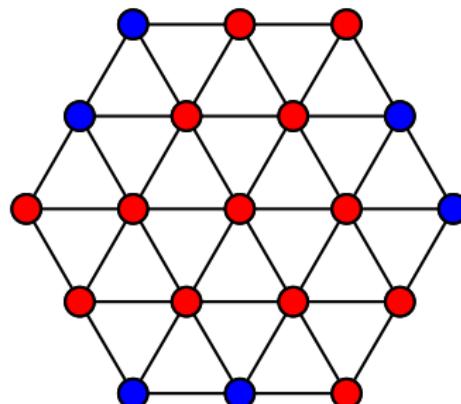
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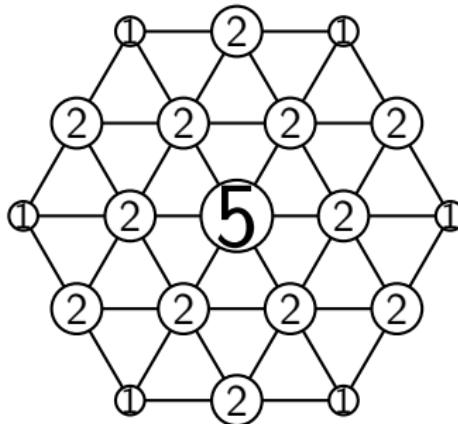


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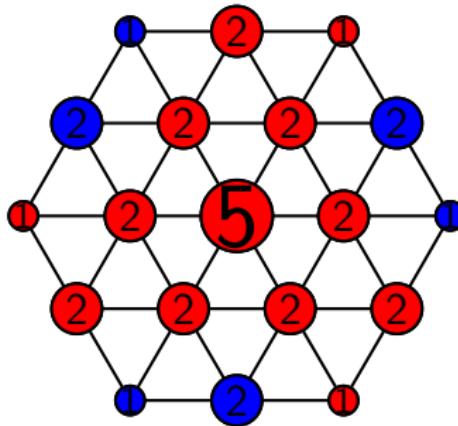


Provided bound : $\frac{6}{19} \simeq 0.316$.

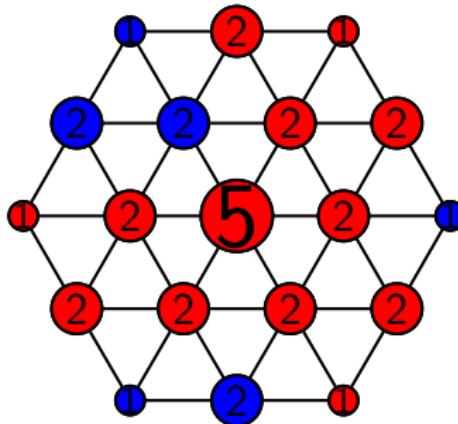
With a weighted graph



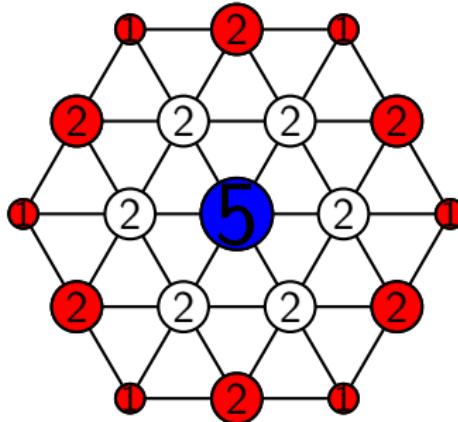
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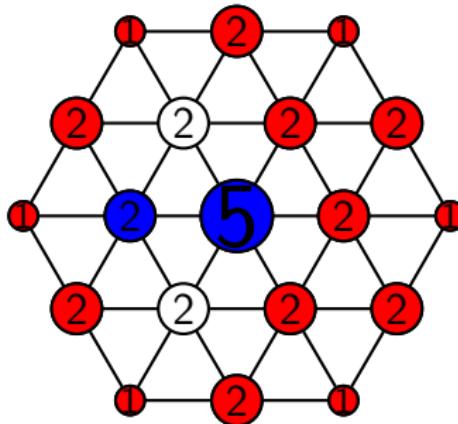
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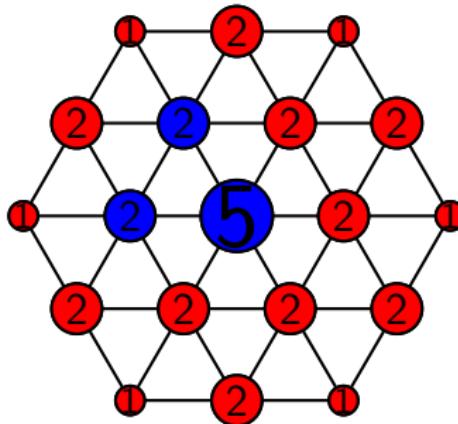
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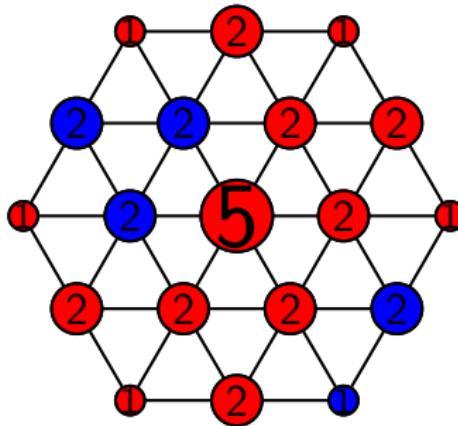
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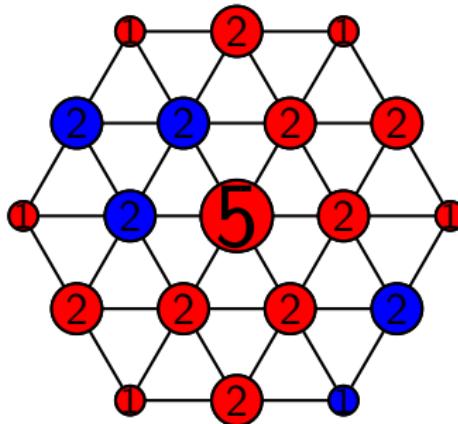
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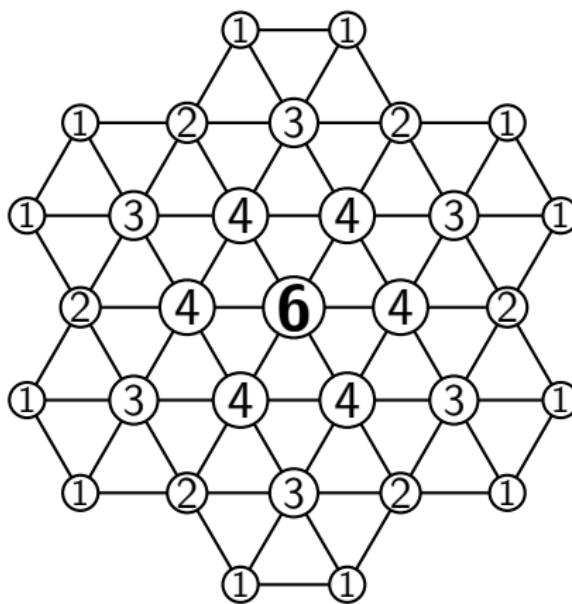
With a weighted graph



With a weighted graph



Provided bound : $\frac{9}{35} \simeq 0.257$.

Alternative proof of $m_1(\mathbb{R}^2, \|\cdot\|_{\mathcal{H}}) \leq \frac{1}{4}$ 

This graph has weighted independence ratio $\frac{1}{4}$.

Fractional colouring

Chromatic number

The chromatic number χ of a graph G is the smallest number a such that a colours are sufficient to colour each vertex of G in such a way that no two adjacent vertices share the same colour.

Fractional colouring

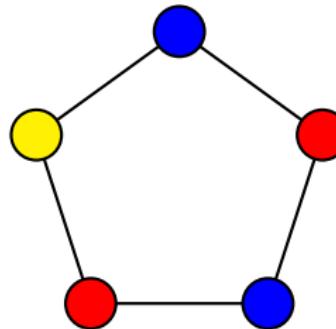
Fractional chromatic number

The **fractional** chromatic number χ_f of a graph G is the smallest number $\frac{a}{b}$ such that a colours are sufficient to **assign b colours** to each vertex of G in such a way that no two adjacent vertices share a common colour.

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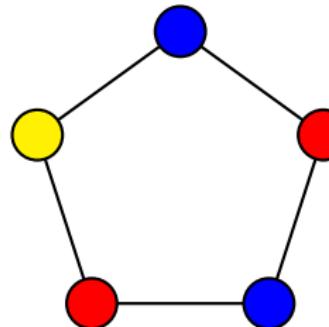


$$\chi(C_5) = 3$$

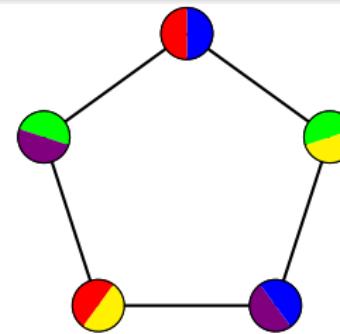
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$$\chi(C_5) = 3$$



$$\chi_f(C_5) = \frac{5}{2}$$

Relation to fractional colouring

Relation between these parameters

$$\alpha^*(G) = \frac{1}{\chi_f(G)}$$

$$\frac{1}{\chi_m(\mathbb{R}^n, \|\cdot\|)} \leq m_1(\mathbb{R}^n, \|\cdot\|) \leq \frac{1}{\chi_f(\mathbb{R}^n, \|\cdot\|)}.$$

Algorithms

- Design of an efficient algorithm for the fractional chromatic number, based on linear programming, especially optimized for our geometric instances.
- Method to build graphs of high fractional chromatic number and of size within the limit of our computational power.

The Euclidean plane

Cranston, Rabern (2017): $\chi_f(\mathbb{R}^2) \geq \frac{76}{21} \geq 3.61904$.
 $\Rightarrow m_1(\mathbb{R}^2) \leq 0.276316$.

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Exoo, Ismailescu (unpublished): $\chi_f(\mathbb{R}^2) \geq \frac{383}{102} \geq 3.75491$.

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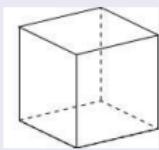
Theorem (Bellitto, Pêcher, Sedillot)

$$\chi_f(\mathbb{R}^2) \geq 3.8977.$$

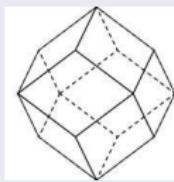
$$m_1(\mathbb{R}^2) \leq 0.25656.$$

Regular 3-dimensional parallelotahedra

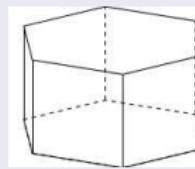
Current results (Bachoc, Bellitto, Moustrou, Pêcher)



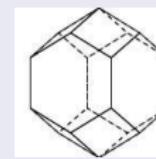
$$m_1 = \frac{1}{8}$$



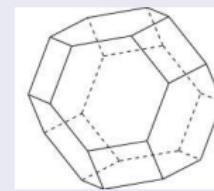
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$$m_1 \leq 0.130443$$

Thank you!