The smallest 5-chromatic tournament

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Joint work with
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Théo Pierron, Université Lyon 1, LIRIS, France

Ramsey's problem

Complete graph where every edge is either red or blue.

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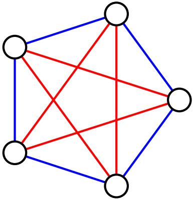

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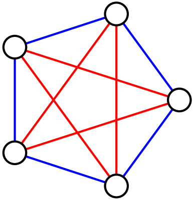

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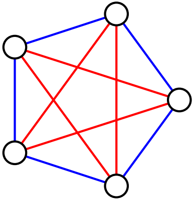

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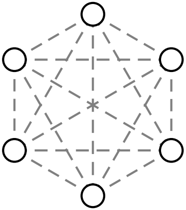

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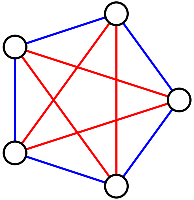

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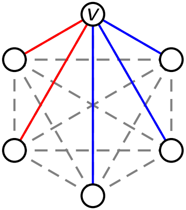

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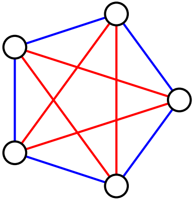

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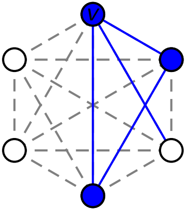

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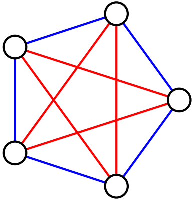

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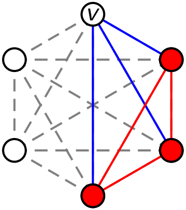

6 vertices

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- Blue and red edges can be replaced by edges and non-edges (every graph with 6 vertices either has a clique or an independant set of size 3).
- Ramsey theory: if a structure is big enough, it must have certain properties.

Ramsey numbers

- $R_1 = 1 \quad R_2 = 2$

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Erdős (reported by Spencer in 1990)

"Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack."

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- $R_5 \leqslant 48$ (Angeltveit and McKay, 2017)
- $R_5 \leqslant 46$ (Angeltveit and McKay, 2024)

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- Probabilistic method (Erdős, 1947):
  If every edge is blue or red with proba $1/2$.
  A 5-clique has 10 edges, it is monochromatic with probability $1/2^9 = 1/512$.
  A graph with 11 vertices has $\binom{11}{5} = 462$ cliques of size 5.
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- Graph enumeration, graph isomorphism problem, automorphism detection (NAUTY, McKay)

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- $R_{a,b,c}$. $R_{3,3,3} = 17$
- Forbidding other structures than cliques.
- Directed Ramsey numbers: edges are oriented instead of being blue and red. We look for ordered sets instead of monochromatic cliques.

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A tournament is transitive iff for every arc uv and vw, there exists an arc uw.
Transitive tournament = Acyclic tournament

Directed Ramsey Numbers

Theorem (Erdős, Moser, 1964)

For all k, there exists $\vec{R}_k$ such that every tournament with at least $\vec{R}_k$ vertices admits a transitive set of size k.

- $\vec{R}_1 = 1$
- $\vec{R}_2 = 2$
- $\vec{R}_3 = 4$
- $\vec{R}_4 = 8$
- $\vec{R}_5 = 14$
- $\vec{R}_6 = 28$ (Sanchez-Flores, 1994)
- $34 \leqslant \vec{R}_7 \leqslant 47$ (Neiman, Mackey, Heule, 2020)
- $\vec{R}_n \in \Theta(2^n)$ (Erdős, Moser, 1964)

Paley tournaments

If $n = 4k + 3$ is prime, the Paley tournament on n vertices P_n is the tournament such that there is an arc from i to j iff $j - i$ is a square $\mod n$.

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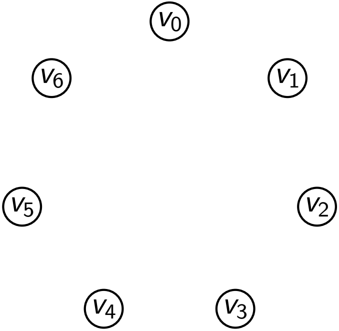

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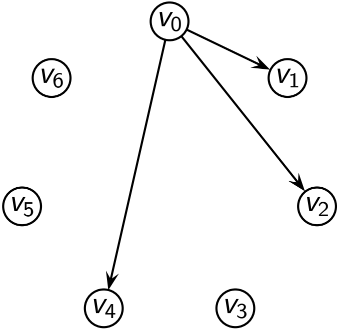

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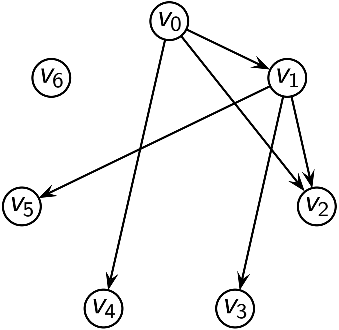

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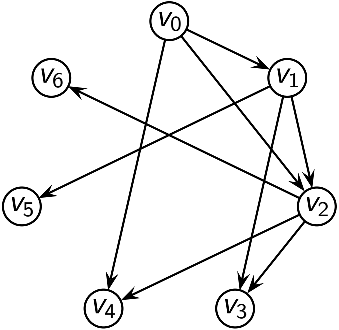

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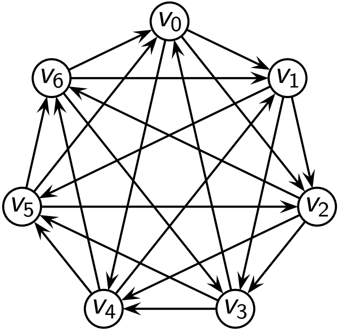

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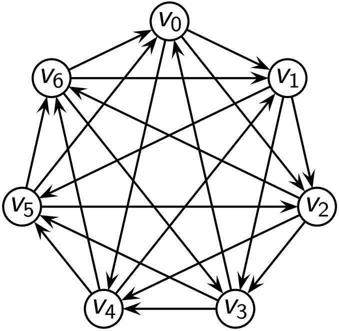

$\vec{R}_4 \geqslant 8$

Directed coloring

Chromatic number

The chromatic number of a graph G is the smallest number of colors required to assign a color to each vertex of the graph so that no color class contains an edge.

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Directed chromatic number (Neumann-Lara, 1982)

The directed chromatic number of a digraph D is the smallest number of colors required to assign a color to each vertex of the graph so that no color class contains a closed walk.

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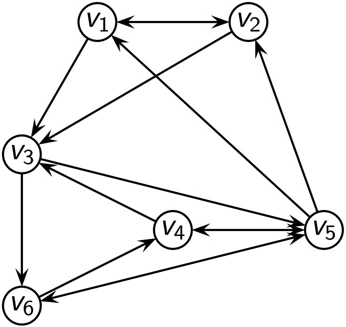

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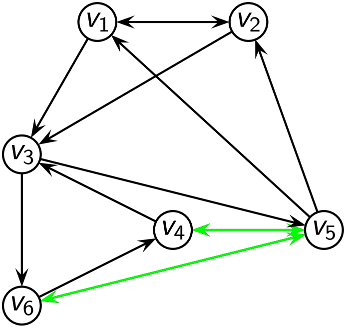

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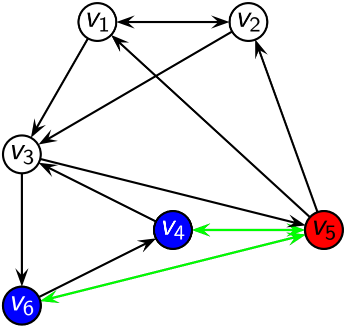

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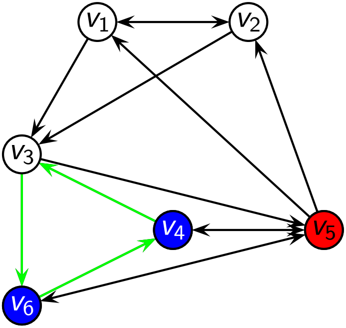

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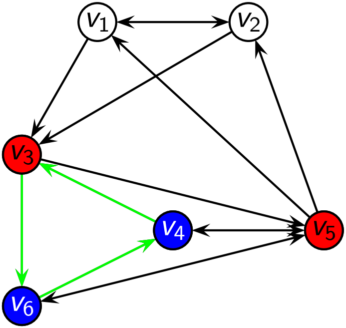

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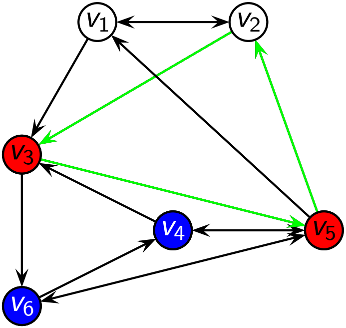

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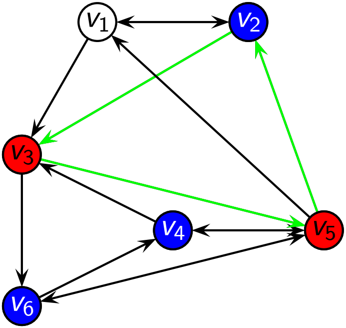

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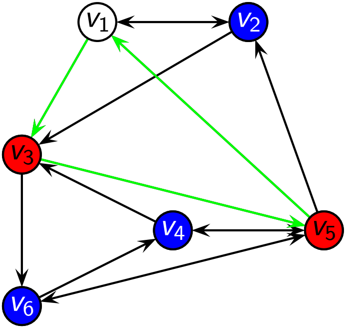

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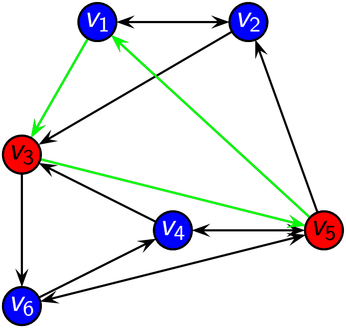

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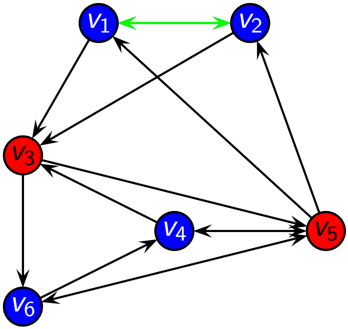

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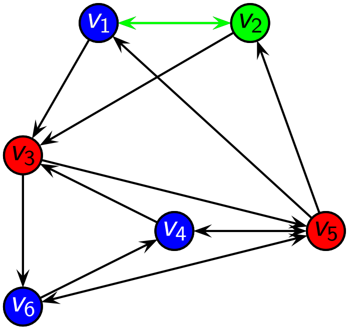

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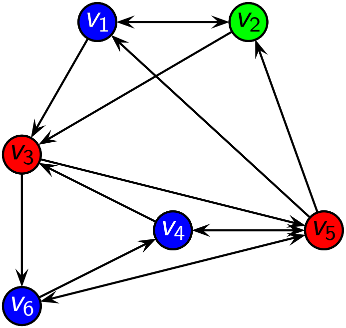

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What is the size n_k of the smallest oriented graph of chromatic number k?

- Can be restricted to tournaments.
- Proper coloring of a tournament = partition into transitive subtournaments.
- Size of the smallest undirected triangle-free graph of chromatic number k? 11 for $k = 4$ (Chvátal, 1970), 22 for $k = 5$ (Jensen, Royle, 1995), open for $k \geqslant 6$, between 32 and 40 (Goedgebeur, 2020).

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Conjecture (Neumann-Lara, 1994)

$$n_5 = 17$$

"I know that $17 \leqslant n_5 \leqslant 19$."

Can we brutefoce it?

Up to isomorphisms, there are

- 244912778438520759443245824 (27 digits) tournaments on 17 vertices;
- 1783398846284777975419600287232 (31 digits) tournaments on 18 vertices;
- 24605641171260376770598003978281472 (35 digits) tournaments on 19 vertices.

Enumerating them up to isomorphisms is difficult.
We have to solve an NP-complete problem on each of them.

1 Introduction: Ramsey theory

2 Our problem

3 Our results

- Tournaments on 12 vertices
- Tournaments on 17 vertices
- Tournaments on 18 vertices
- Tournaments on 19 vertices

Structure

Theorem (Sanchez-Flores, 1998)

There is a unique tournament on 12 vertices that does not contain a TT_5 and it is 3-chromatic.

Consequence

In every 4-chromatic tournament on 12 vertices, there is a TT_5 whose removal yields one of the four 3-chromatic tournament on 7 vertices.

Results

Theorem (Bellitto, Bousquet, Kabela, Pierron)

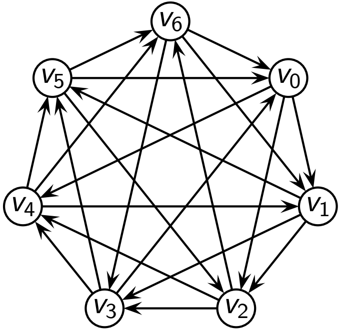
Every 4-chromatic tournament on 12 vertices :

- contains P_{11};
- is a junction of TT_5 and W_1 (the 3-chromatic 7-vertex tournament contained by P_{11}).

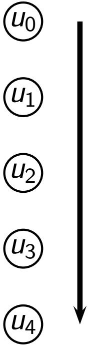
There are 3-chromatic tournaments on 8 vertices that do not contain any 3-chromatic tournaments on 7 vertices.

Computer assisted proof

P_7



TT_5



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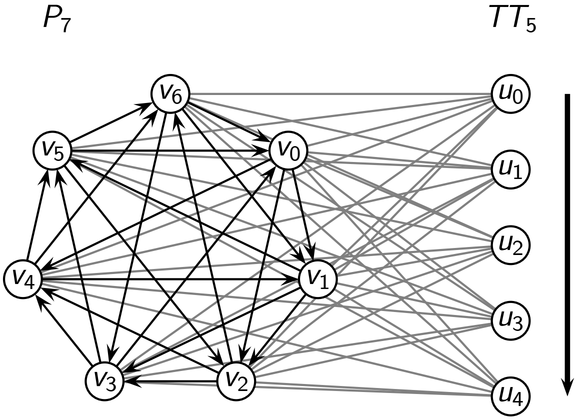

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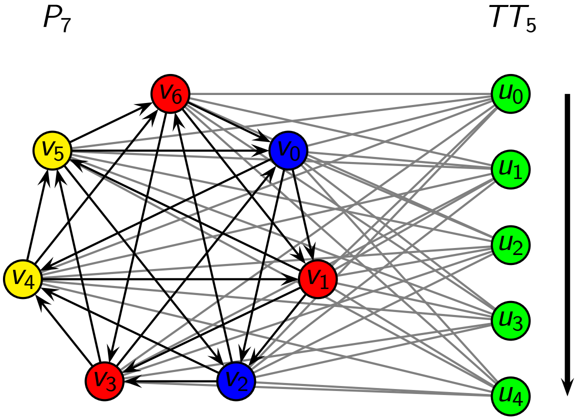

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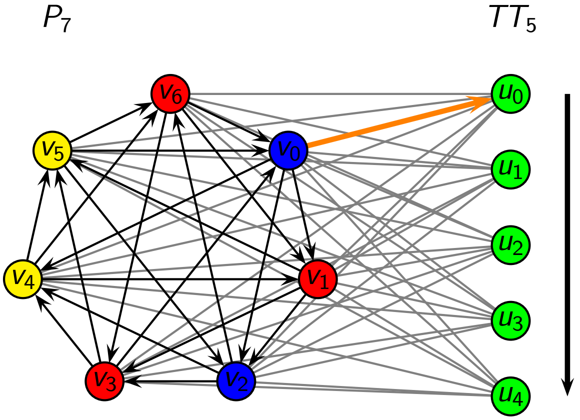

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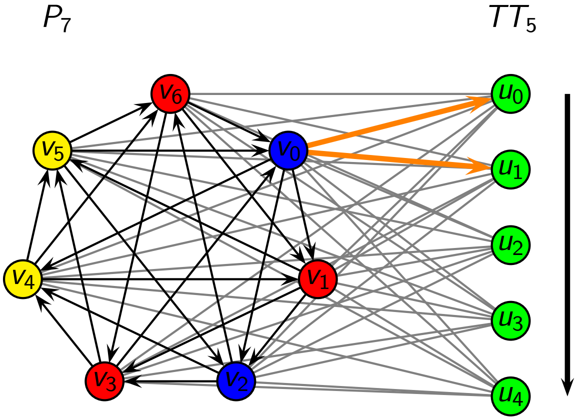

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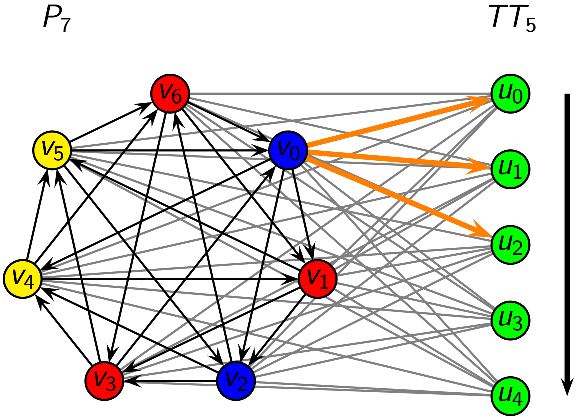

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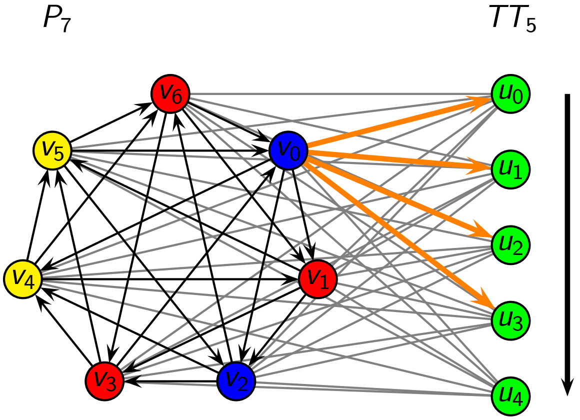

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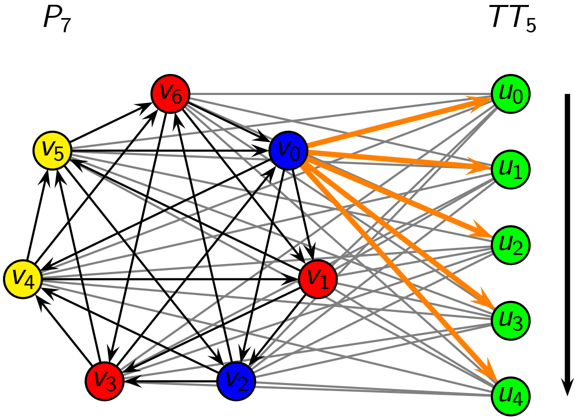

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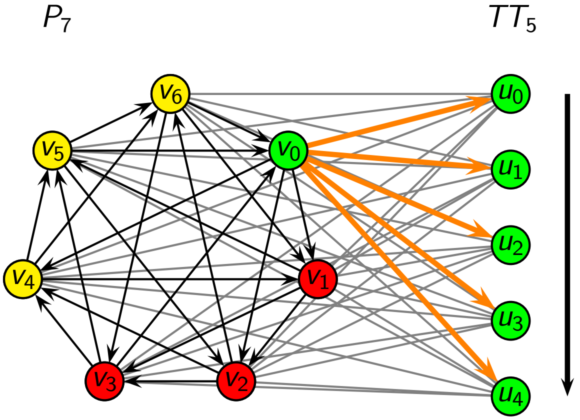

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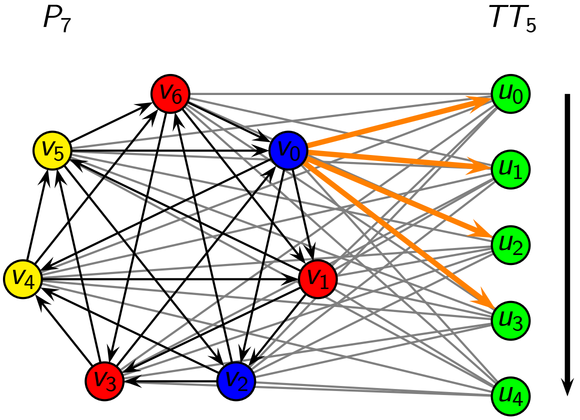

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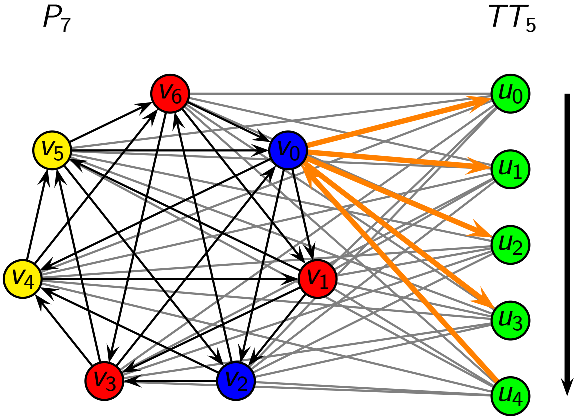

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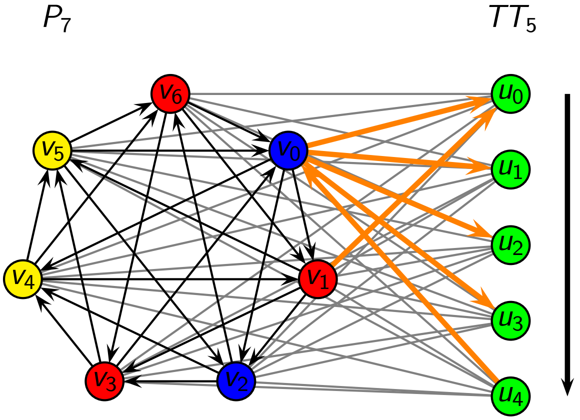

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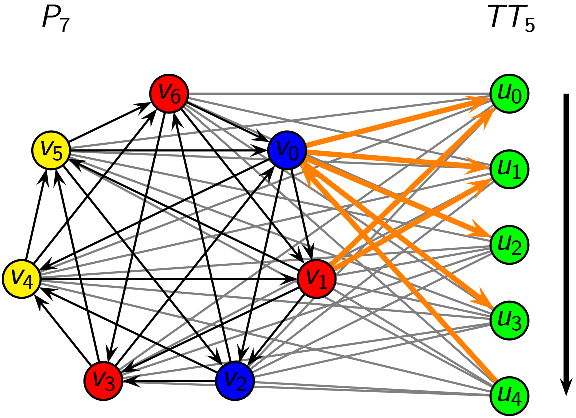

Outline of the proof

Structure of the graph

If T is a 5-chromatic tournament on 17 vertices, then we can partition its vertices into A_1, A_2 and B such that

- A_1 and A_2 induce two copies of TT_5
- B induces a copy of W_1

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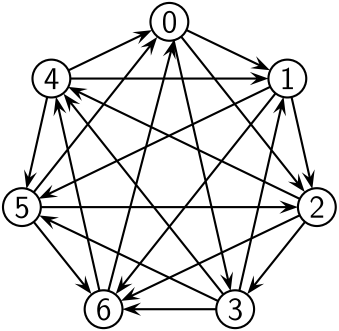
41 arcs decided, 95 left to go...

Idea

Can we partition B into B_1 and B_2 such that the tournaments induced by $A_1 \cup B_1$ and $A_2 \cup B_2$ are 2-colorable?



- $\chi(A_i \cup \{0, 1, 4\}) = 2.$
- $\chi(A_i \cup \{0, 1, 2, 3\}) = 2$
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- If $\chi(A_i \cup \{4, 5, 6\}) > 2$ and $\chi(A_i \cup \{2, 3, 5, 6\}) > 2$, then $\chi(A_i \cup \{0, 2, 4\}) = \chi(A_i \cup \{1, 3, 5, 6\}) = 2.$
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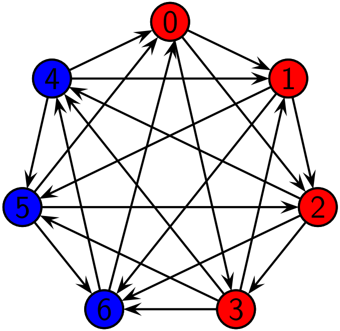

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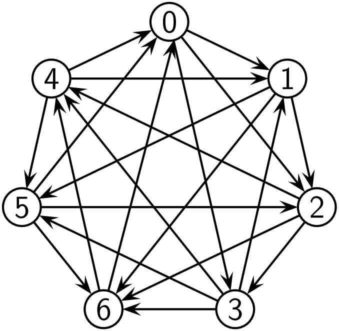

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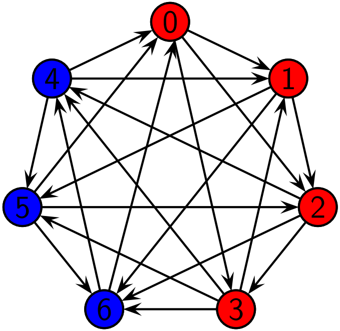

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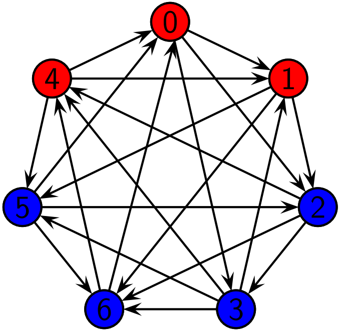

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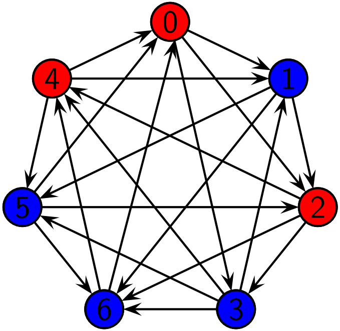

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Combines ideas from the previous sections

Let T be 5-chromatic on 18 vertices.

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Let T be 5-chromatic on 18 vertices. If T has 3 disjoint TT_5

- We build all the 3-chromatic 8-vertex tournaments we can by joining C_3 and TT_5.
- We build all the 4-chromatic 13-vertex tournaments we can by joining C_3 and 2 TT_5.
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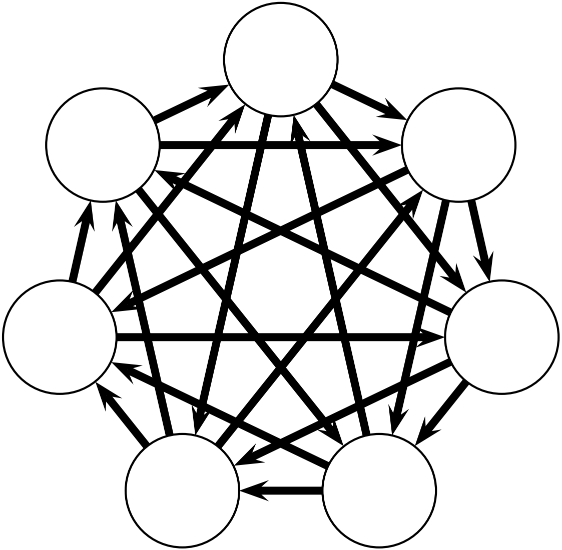
If T has 2 disjoint TT_5

- Same idea as previous section but B induces one of the 94 3-chromatic 8-vertex TT_5-free tournaments.

A 5-chromatic tournament on 19 vertices



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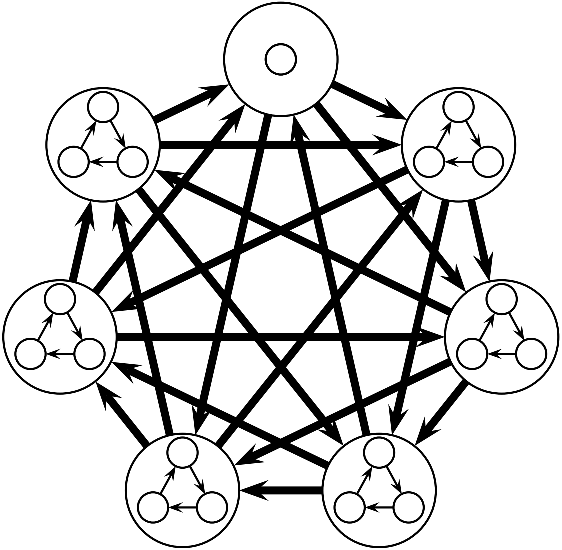

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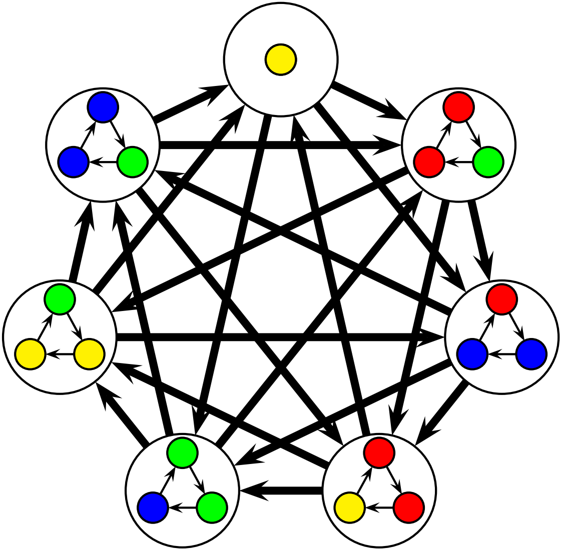

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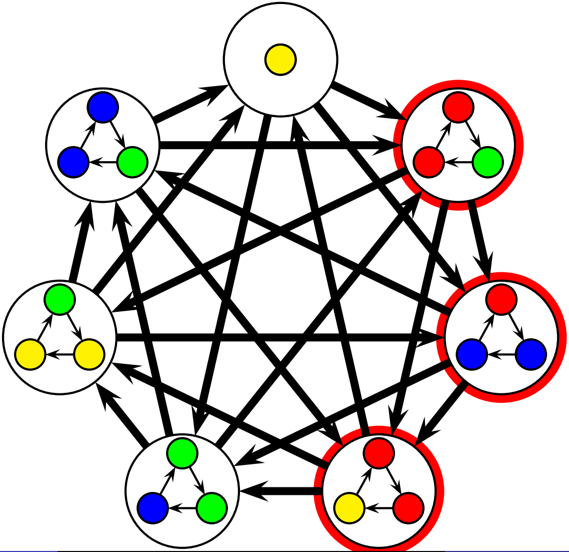

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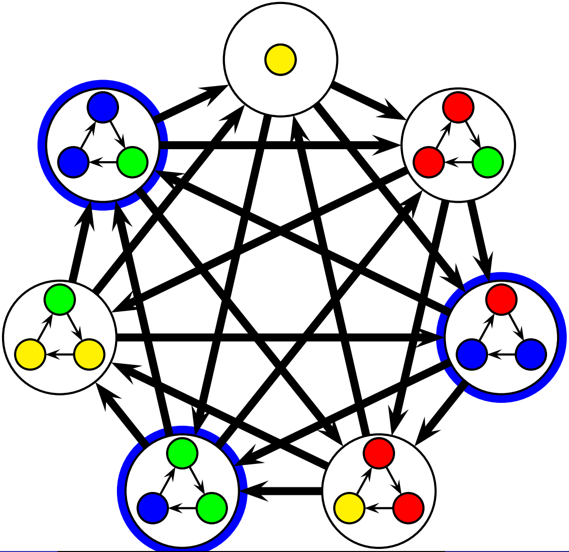

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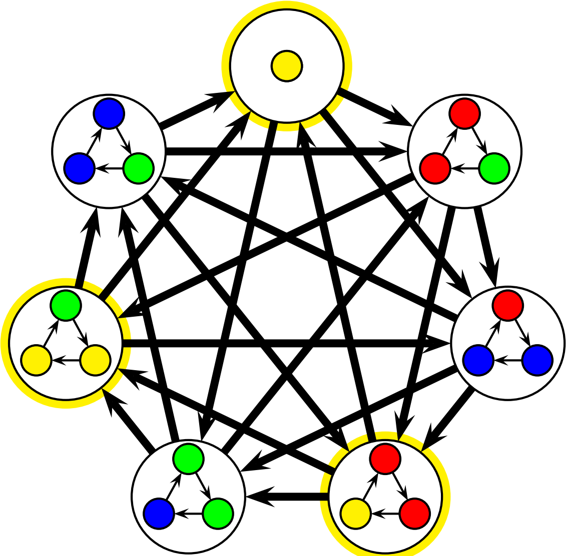

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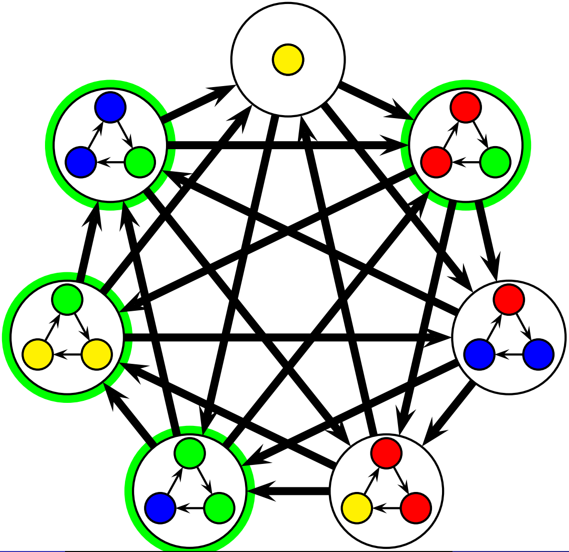

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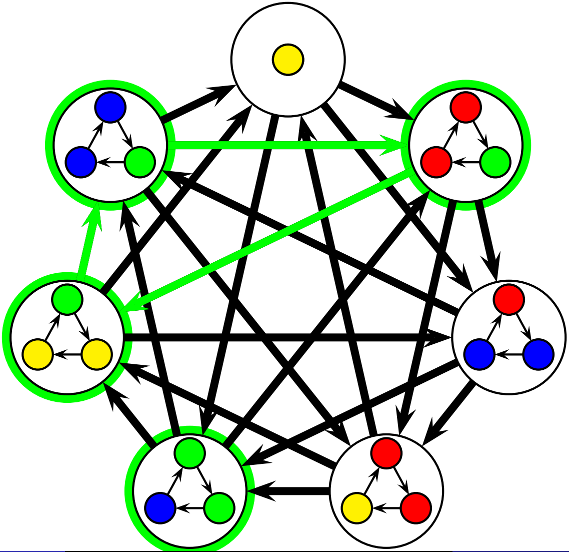

Perspectives

- Elegant proof that there are no vertex-critical tournament on 12 vertices.
- Counting/enumerating the 5-chromatic 19-vertex tournaments?
- n_6 ?

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Thank you!