Restless exploration of periodic temporal graphs

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Our model

Temporal graph

A *temporal graph* is defined by

- a set of vertices V
- a sequence of set of edges $E_1, E_2, \cdots$ that may or may not be finite

The graph $G_t = (V, E_t)$ is called the *snapshot* at time t.

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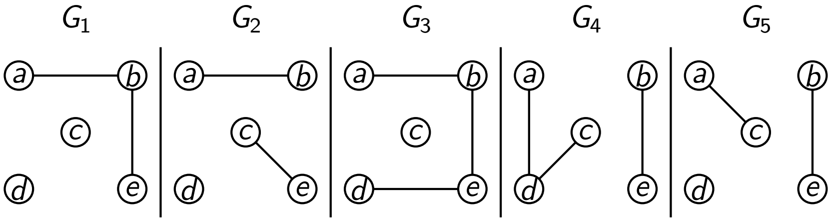

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Definition

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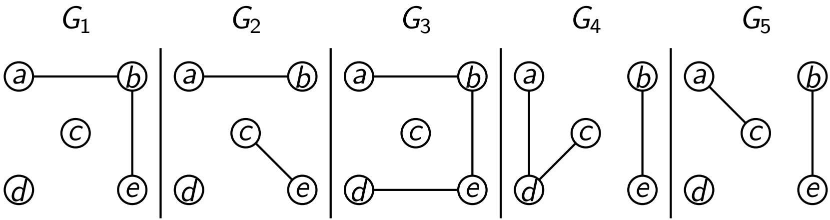

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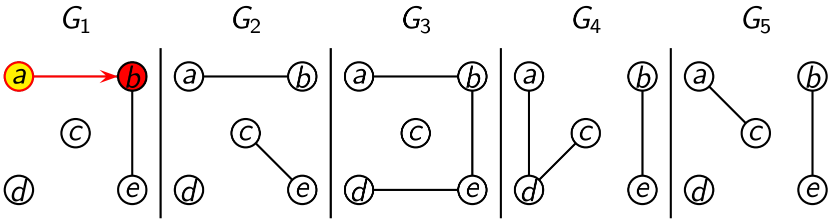

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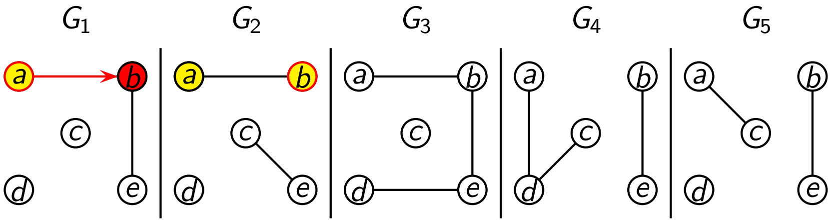

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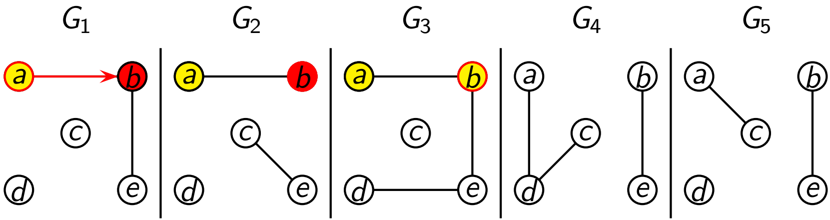

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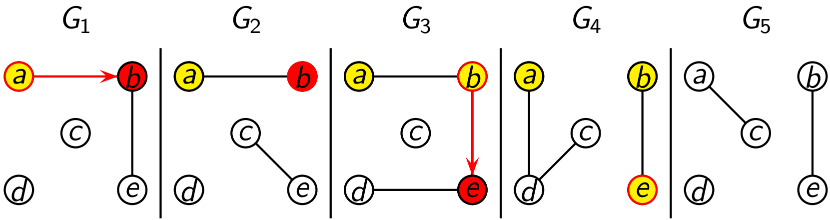

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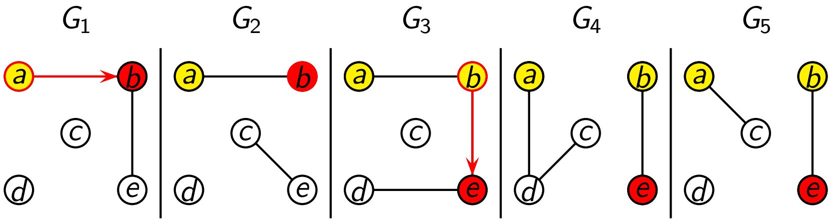

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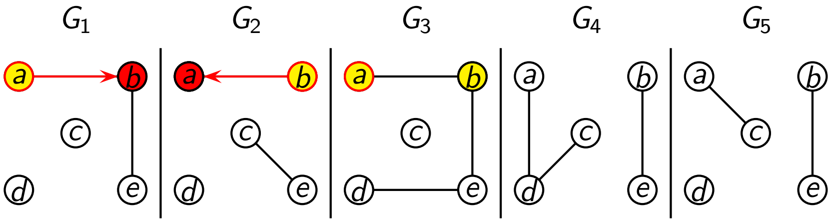

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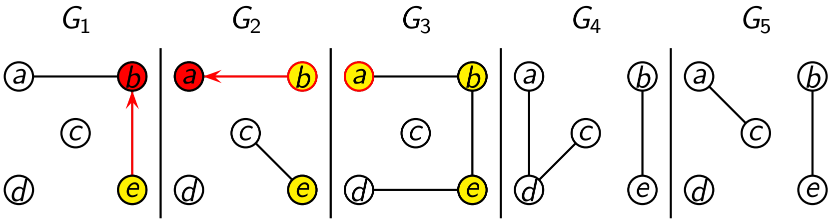

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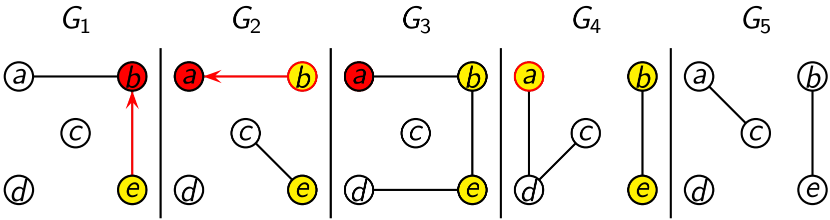

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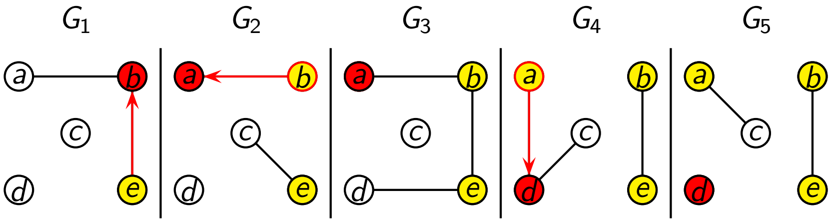

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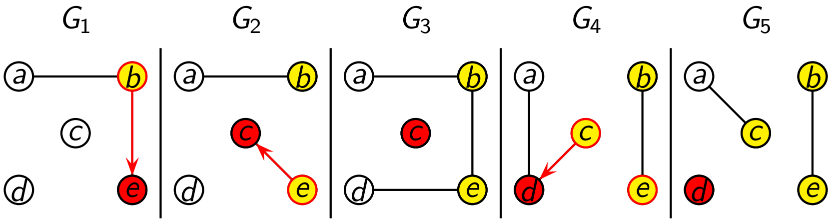

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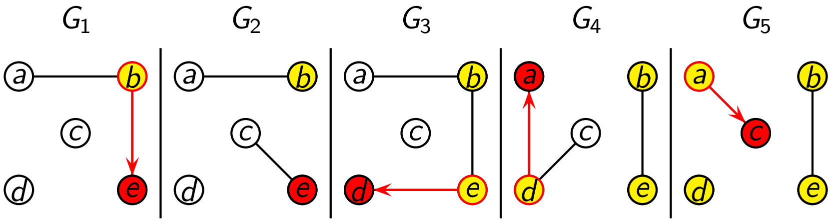

NP-complete in the lifespan of the graph !

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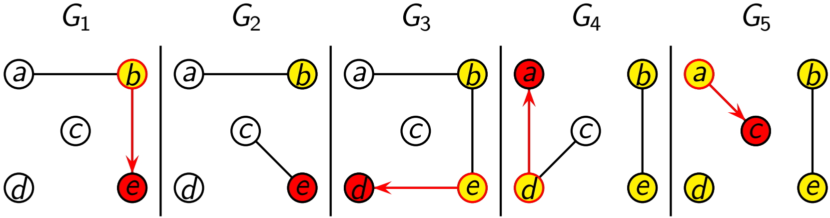

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- If the graph is periodic, possible iff the union of the snapshots is connected.

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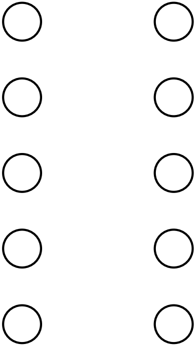
Erlebach, Hoffmann and Kammer's construction



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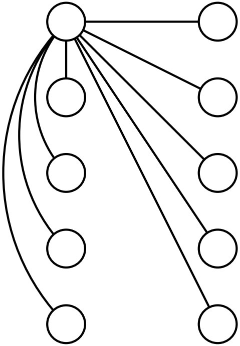

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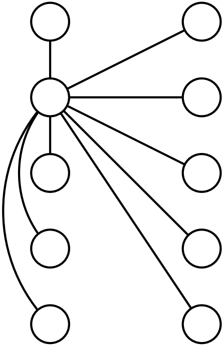

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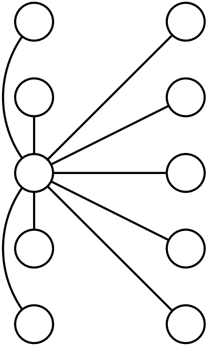

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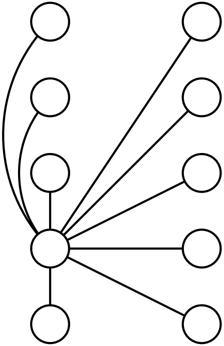

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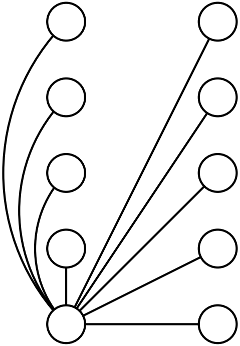

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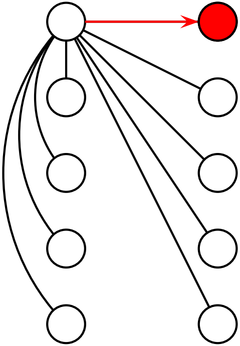

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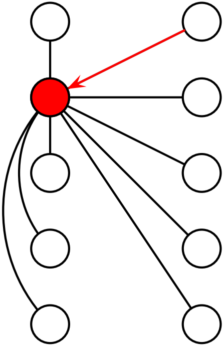

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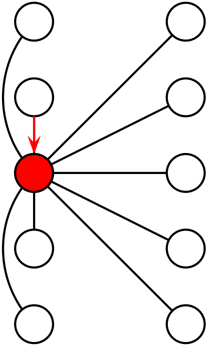

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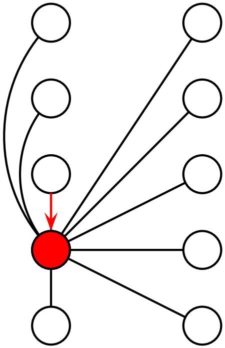

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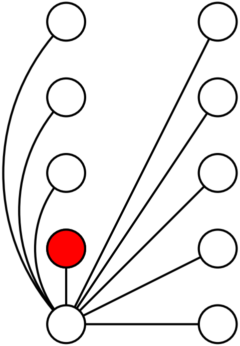

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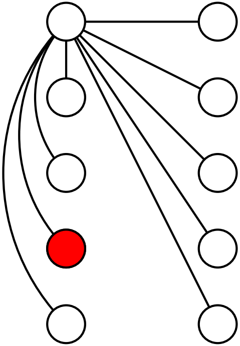

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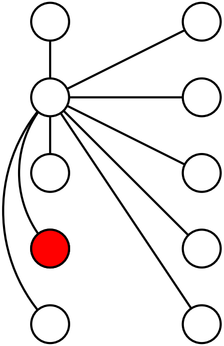

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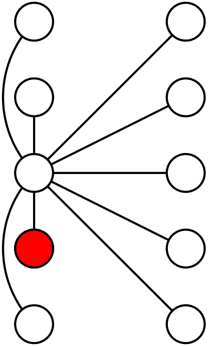

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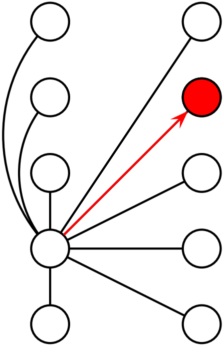

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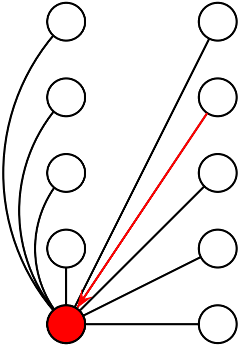

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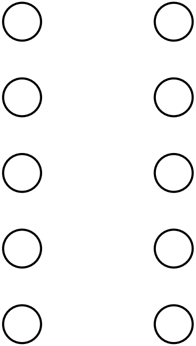

The $O(n^2)$-bound is tight

Restlessness

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A journey is Δ-restless is not allowed to wait Δ steps in a row. For example :

- routing in non delay tolerant network, with memory limitation on the nodes ;
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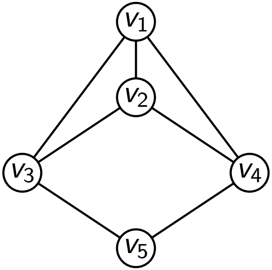
Our problem

Given a periodic temporal graph, can we explore it restlessly ?

Edge-colouring

Let $G = (V, E)$ be an undirected graph.

- k-edge-colouring : function $c : E \mapsto [1, k]$
- proper edge-colouring : $c(uv) \neq c(vw)$.



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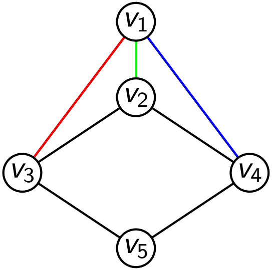

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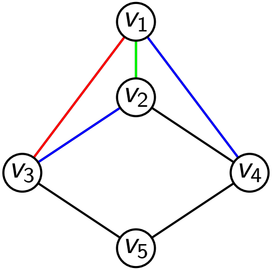

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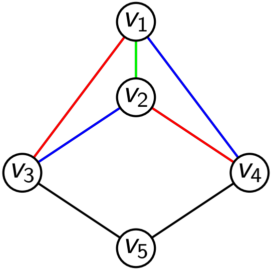

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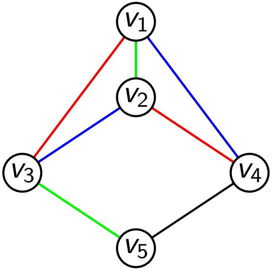

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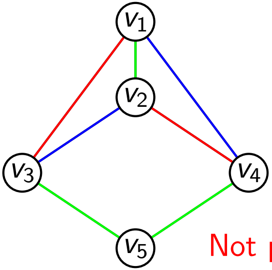

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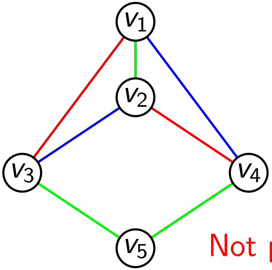

Not proper !

- Properly-coloured walk : does not use consecutively two edges of the same colour.

Context

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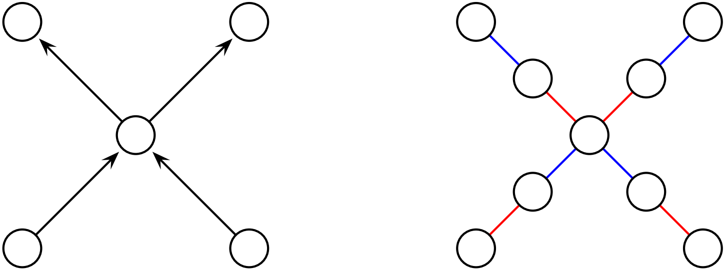
- Powerful model.



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An open problem

If G is a complete k-edge colored multigraph :

- if $k = 2$, we can say in polynomial time if G admits a properly-colored hamiltonian cycle
- if $k \geqslant 3$, the complexity is open
- if the cycle has to be strictly alternating, polynomial for every k

Edge-coloring and temporality

Let $G_t = (V, (E_1, E_2, \cdots, E_k))$ be a k-periodic temporal graph.
Let G_c be a k-edge-colored multigraph with :

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- k-restless journey in G_t = properly-colored path in G_c.
- 1-restless journey in G_t = strictly alternating path in G_c.

Our results

Theorem

One can decide in polynomial time whether a 2-periodic temporal graph can be 1-restlessly explored.

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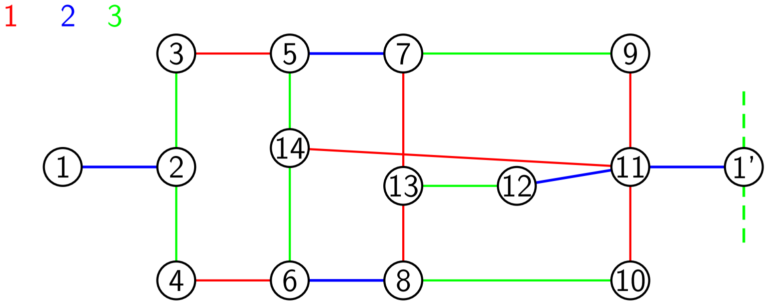
Theorem

For every $p \geqslant 3$, it is NP-complete to decide whether a p-periodic temporal graph can be 1-restlessly explored.

Idea of the proof



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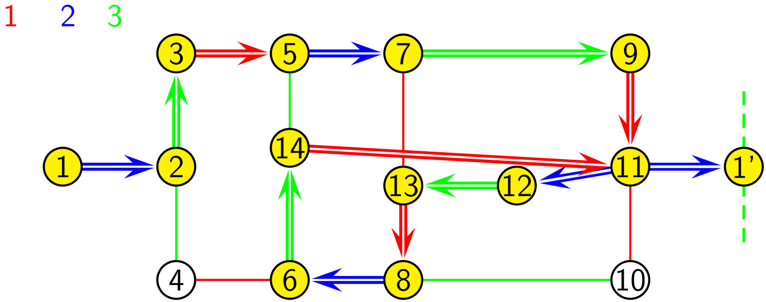
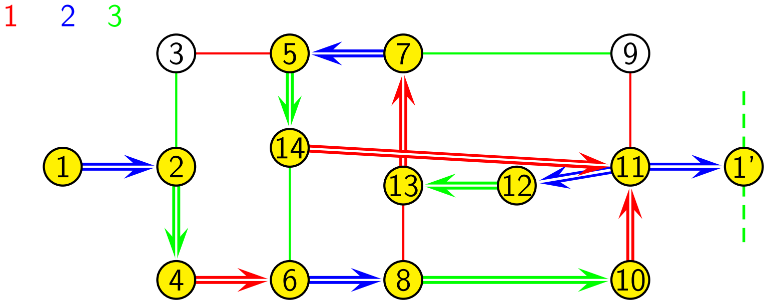


Idea of the proof



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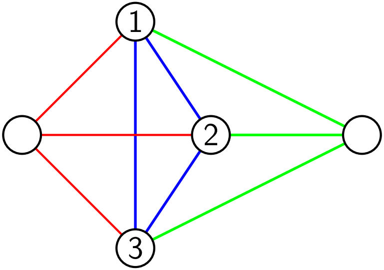

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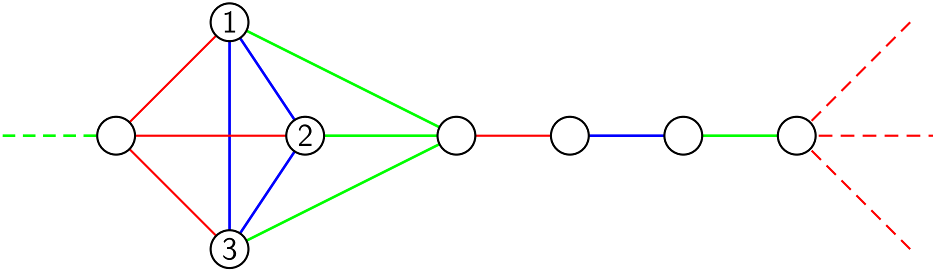

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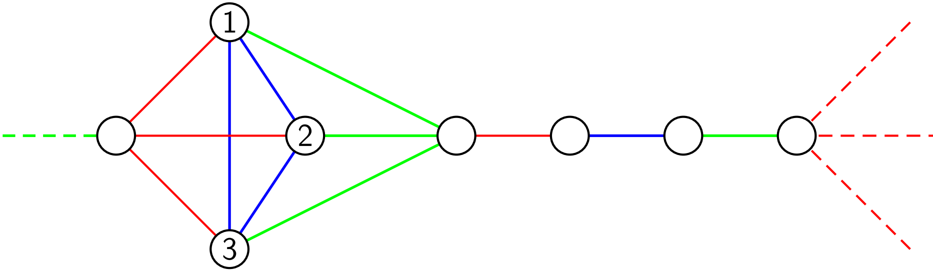

3-SAT-(2,2) : 3-sat where every variable has exactly 2 positive and 2 negative occurrences. NP-complete (Berman, Karpinski, Scott)

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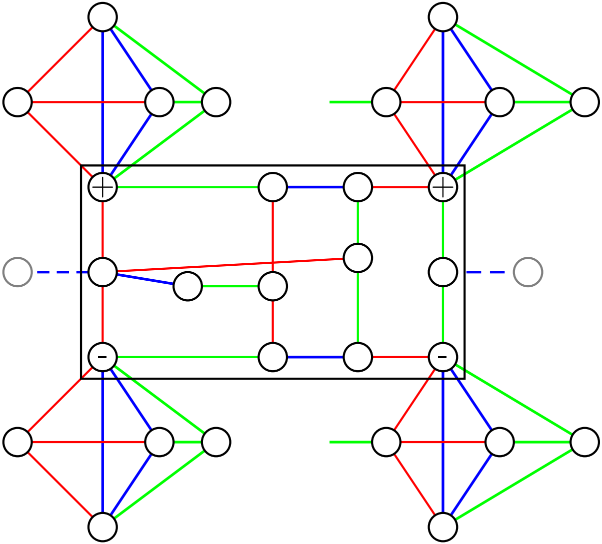

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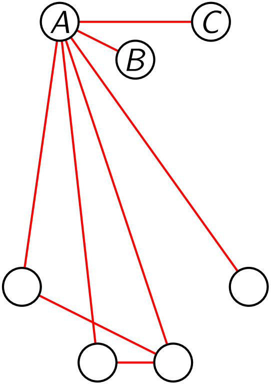

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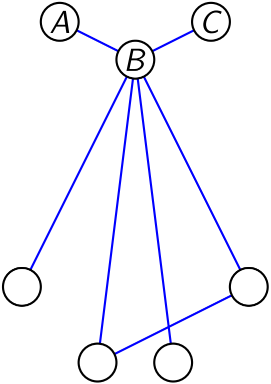

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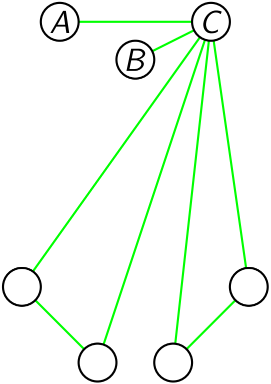

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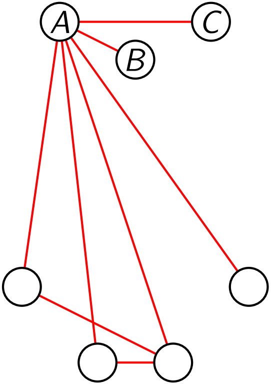

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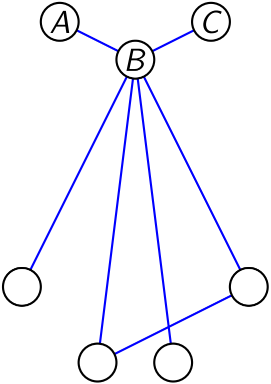

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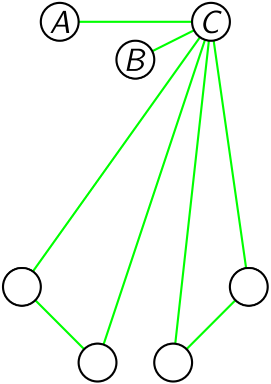


Time bounds

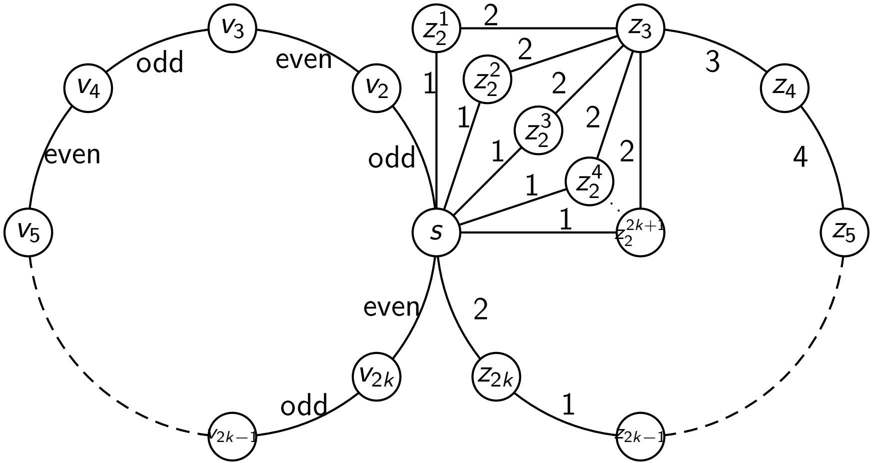
Upper bound

Any explorable p-periodic temporal graph can be explored restlessly in at most pn^2 steps.

Lower bound

For every $p \geq 2$ there are families of explorable p-periodic temporal graphs that require $\Omega(pn^2)$ steps to be explored.

Our construction



Open questions

- Every explorable p-periodic graph can be explored in time pn^2 and some require $\frac{pn^2}{18}$. Can we close the gap ?

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- What about k-restlessness with $k > 1$?
- FPT algorithm for the NP-complete case ?
- What if the graph cannot change too much between two consecutive frames ?

Thank you !