On Minimum Connecting Transition Sets in Graphs

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1. Context
   - Forbidden transitions
   - Connecting transition sets

2. Our results
   - Upper bounds
   - Reformulation
   - Main results

Motivation

We use graphs to model networks in various fields of application (road networks, telecommunication networks, public transit...).

Set of possible walks in a road network $\neq$ set of walks in the graph.

We need a stronger model !

Forbidden-transition graphs

- Transition : pair of adjacent distinct edges.
- Forbidden-transition graph : $G = (V, E, T)$ where T is the set of permitted transitions.
- The walk $W = (v_1, v_2, \ldots, v_k)$ *uses* the transition $v_i v_{i+1} v_{i+2}$ for all i such that $v_i \neq v_{i+2}$.
- The walk W is *T-compatible* if and only if it only uses transitions from T.

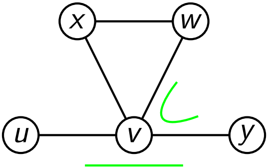
Related works

- Properly-coloured walks in edge-coloured graphs.
- Antidirected walks in digraphs.
- Forbidden subpaths / subwalks.

T-connectivity

T-connectivity

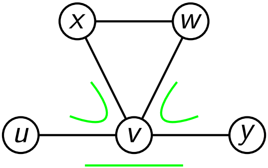
A graph $G = (V, E)$ is T-connected if and only if there exists a T-compatible walk between each pair of vertices.
In this case, we say that T is a connecting transition set of G.



$T_1 = \{uvy, wvy\}$

There exists a T_1-compatible walk between w and every vertex of the graph.

There exists no T_1-compatible walk between x and u or y.



$$T_1 = \{uvy, wvy\}$$

$$T_2 = \{uvy, wvy, uvx\}$$

There exists a T_1-compatible walk between w and every vertex of the graph.

There exists no T_1-compatible walk between x and u or y.

The set T_2 is a connecting transition set of the graph.

Minimum connecting transition set

Minimum connecting transition set

Smallest set T such that a graph G is T-connected.
(similar to minimum spanning trees)

Several papers study robustness of graph properties to malfunctionning transitions.

$\rightarrow$ Application of minimum connecting transition sets to robust network design ?

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2. Our results
   - Upper bounds
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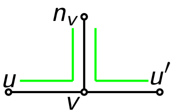
General case

Pick a neighbour n_v for every vertex v.

$$T = \bigcup_{v \in V} \bigcup_{u \in N(v) \setminus \{n_v\}} \{uvn_v\} \text{ is connecting.}$$

$$|T| = 2|E| - |V|$$



If G is a tree, $|T| = n - 2$ and this construction is optimal!

General upper bound

A minimum connecting transition set of a graph of n vertices has size at most $n - 2$.

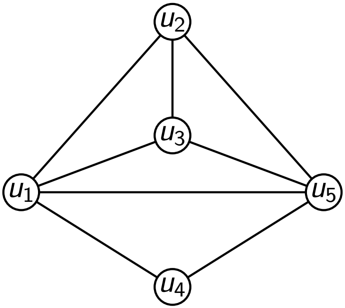
Lower bounds and reachable values

Complete graph : minimum connecting transition set of size 0.

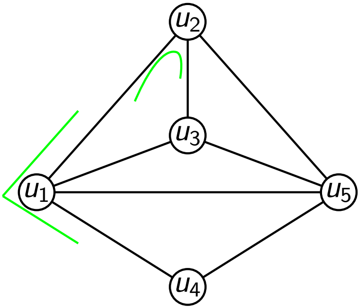
Every value k between 0 and $n-2$ is reachable : start from a tree of size $k+2$ and add $n-k+2$ dominating vertices.

Question

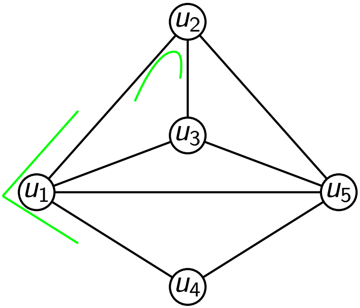
Is the size of a minimum connecting transition set $k-2$ where k is the number of non-dominating vertices ?

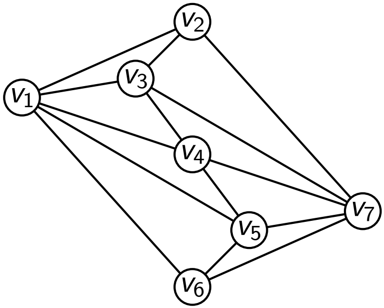


3 non-dominating vertices.



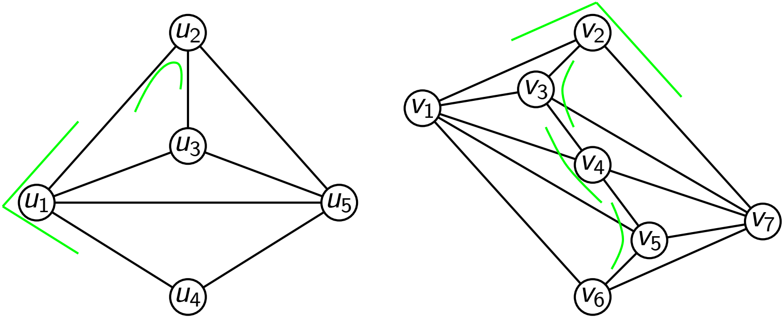
3 non-dominating vertices.
Size of MCTS : 2.





3 non-dominating vertices.
Size of MCTS : 2.

7 non-dominating vertices.



3 non-dominating vertices.
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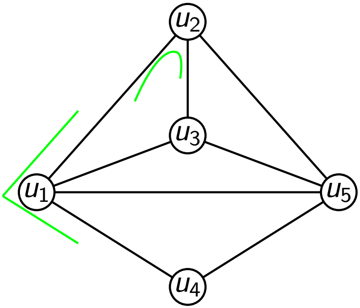
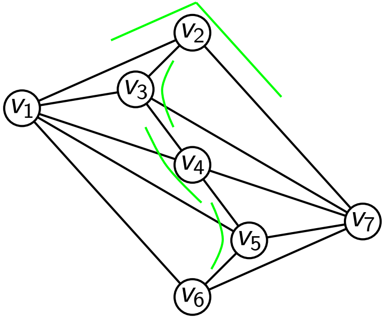
7 non-dominating vertices.
Size of MCTS : 4.





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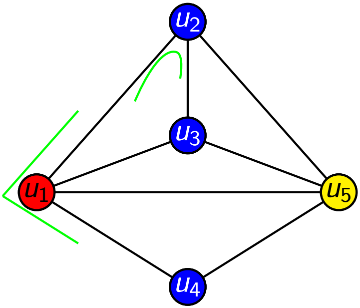
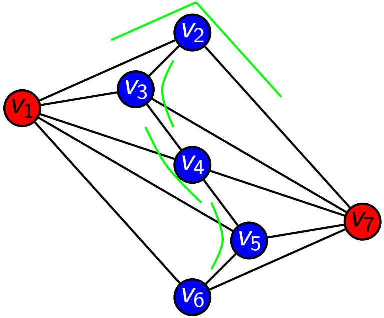
Key

Look at the connected components of the complementary graph ! (co-cc)





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Key

Look at the connected components of the complementary graph ! (co-cc)

Improved upper bound

Theorem

Every connected graph G has a connecting transition set of size $\tau(G)$ where

$$\tau(G) = \sum_{\substack{C \text{ co-cc of } G \\ |C| \geqslant 2}} \begin{cases} |C| - 2 \text{ if } G[C] \text{ is connected} \\ |C| - 1 \text{ otherwise} \end{cases}$$

Co-P_7

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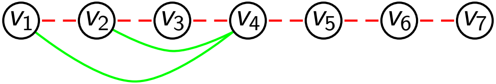

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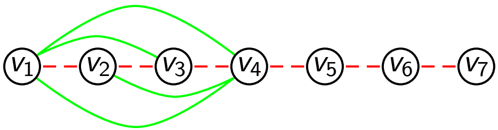

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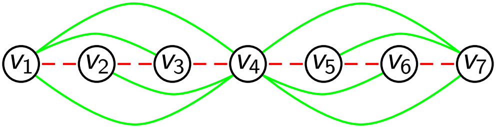

Connecting transition set of size 4 but $\tau(\overline{P_7}) = 5$.

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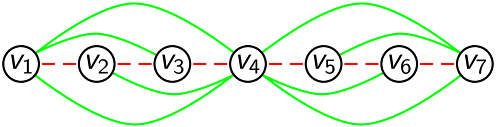

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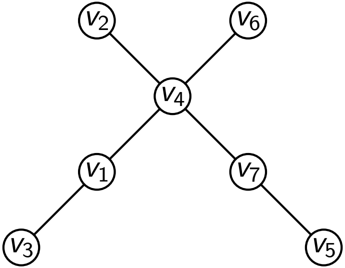

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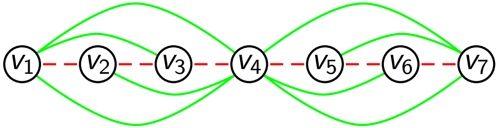

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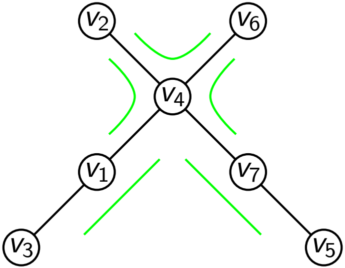

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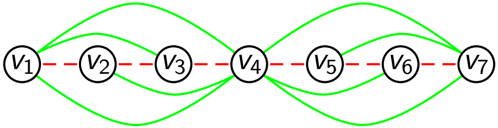

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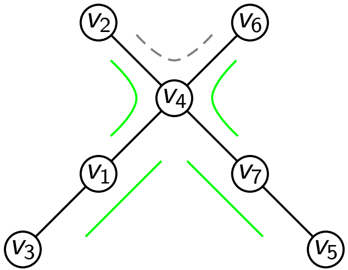

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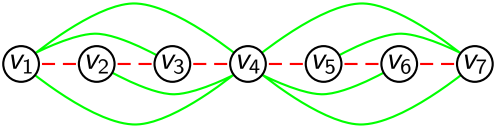

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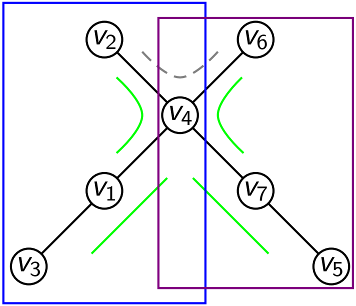

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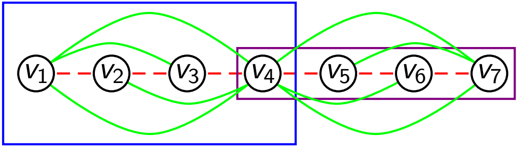

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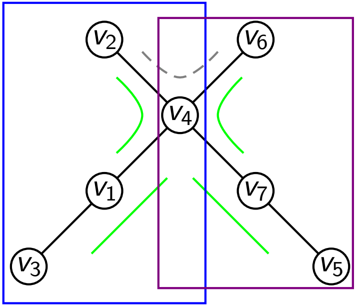

Optimal connecting hypergraph

Connecting hypergraph :

A *connecting hypergraph* of a connected graph G is a set H of subsets of $V(G)$ called *connecting hyperedges*, such that

- For all $e \in H$, $|e| \geqslant 2$.
- For all $e \in H$, $G[e]$ is connected.
- For all $uv \notin E(G)$, there exists $e \in H$ such that $u, v \in e$

$$\text{cost}(H) = \sum_{e \in H} (|e| - 2)$$

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Finding a minimum connecting transition set is equivalent to finding connecting hypergraph of minimum cost !

Complexity and approximation

Complexity

Finding a minimum connecting transition set of a graph is NP-complete.

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Finding a minimum connecting transition set of a graph is NP-complete.

Approximation

The size of a minimum connecting transition of a graph G is at least $\frac{2}{3}\tau(G)$ (tight bound) where

$$\tau(G) = \sum_{\substack{C \text{ co-cc of } G \\ |C| \geqslant 2}} \begin{cases} |C| - 2 \text{ if } G[C] \text{ is connected} \\ |C| - 1 \text{ otherwise} \end{cases}$$

$\rightarrow$ We have a $O(|V|^2)$ $\frac{3}{2}$-approximation !

Our results

- Bounds or exact results for several families of graphs (trees, graphs with cut-vertices).
- Reformulation of the problem.
- Polynomial $\frac{3}{2}$-approximation.
- Proof of NP-completeness (even when restricted to co-planar graphs).

Future works

- Sparse graphs (bounded treewidth ? bounded maximum average degree ? planar ?).
- Stretch of the minimum connecting sets.
- Starting with forbidden transitions.
- Directed graphs.

Thank you !